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LATTICE THEORY

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PREFACE

It is now nearly fifty years since the publication of Schröder's monumental *Algebra der Logik*. Since then, the main developments in mathematical logic have been along lines remote from Boolean algebra, while the main developments in Boolean algebra have been in the direction of applications having little to do with logic. In rapid succession, modern algebra, projective geometry, set theory, functional analysis, and probability have been shown to depend fundamentally on generalizations of the "algebra of logic," as understood by Schröder. Besides, there has been so great an increase in general algebraic technique that application of this technique to Boolean algebra has yielded a wealth of new results.

These facts make it desirable to have available a book on the "algebra of logic," written from the standpoint of algebra rather than of logic. Such a book I have attempted to write. A feature of this task which has been especially attractive to me has been fitting into a single pattern ideas developed independently by mathematicians with diverse interests.

This pattern consists in a skeleton of theory developed with examples in the opening sections of the first seven chapters,* which is complete in itself except for the assumption of a passing acquaintance with modern algebra. These opening sections introduce in each successive chapter a more special kind of lattice, whose detailed properties are discussed at length in the remainder of the chapter.

Into this pattern, the various applications of lattice theory fit in a rather unpredictable manner.

The cumulative work of Dedekind, the school of Emmy Noether, and Ore, on the structure of algebras, falls mainly into Chapters II-III.† The applications to topological algebra and set theory come partly before and partly after this material.‡ The applications to functional analysis, due in large part to Kantorovitch, occupy Chapter VII.

Likewise, the logical tradition of Boole, Peirce and Schröder, as expanded today by Stone and Tarski, occupies Chapters VI and VIII.§ The material relevant to projective geometry comes in Chapter IV, although the material on topological algebra preceding it is needed for von Neumann's theory of "continuous geometries." Similarly, the theory of probability developed in Chapter IX is based on parts of Chapters VII and VIII.

* More precisely, §§1-10, the first three or four sections of Chapters II-VII, and §§26-7.

† More precisely, §§19-33, Chapter III omitting §§51-3, and Chapters IV-V passim (esp. §§73-7).

‡ Cf. Chapter I, Chapter II omitting §§30-3 and §§39-40, §§51-5 of Chapter III, §§101-2 of Chapter V, and Chapter VI.

§ More precisely, Chapter VI through §117. The material in the Foreword might also interest the specialist in mathematical logic.

I have tried to acknowledge in the text the fundamental contemporary work of Ore, von Neumann, Stone, and Kantorovitch—each of whom has added a whole chapter to this volume,—and of numerous other contributors. I wish to take this opportunity of recognizing more informal obligations.

It is a pleasure to recall the friendly stimulus which I received from my father and Philip Hall in the earlier stages of this work, and from John von Neumann, Marshall Stone, and Stanislas Ulam in its later stages. And I owe much to the Society of Fellows of Harvard University for its encouragement during three years of complete freedom, when much of the present material was gathered.

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FOREWORD ON TOPOLOGY AND ABSTRACT ALGEBRA

In order to understand pure lattice theory, the only preliminary knowledge required is a grasp of such basic concepts as those of class, element, relation, correspondence, function, and sequence. These concepts are all logical, rather than mathematical.

But in order to appreciate the applications of lattice theory to other branches of mathematics, one should also be reasonably conversant with the fundamental notions of topology and abstract algebra. A familiarity with these notions also makes lattice theory much more vivid, and enables one to make correlations between the ideas of lattice theory and ideas with which every mathematician is acquainted.

Accordingly, those definitions of set theory and algebra which are most frequently used in the sequel are collected in the following paragraphs for convenience.

It will be assumed that the reader is familiar with the notions of the sum (or union), the product (or intersection), and the complement of subsets of a class.* In terms of these, one can easily define a "ring" of sets as a family of subsets of a class which contains with any two subsets, their sum and product. A "field" of sets is defined as a ring of sets which contains with every set its complement.

A ring of sets is called a "σ-ring," if and only if it contains the sum and product of any sequence (i.e., countable set) of its members; a σ-ring which is a field is called a "σ-field" of sets.

The reader should also be familiar with Fréchet's† notion of a "metric space." By this is meant a class of elements (points), together with a definition of the "distance" $\delta(x, y)$ between all pairs of points, the latter being assumed to satisfy four conditions: (1) $\delta(x, x) = 0$, (2) if $x \neq y$, then $\delta(x, y) > 0$, (3) $\delta(x, y) = \delta(y, x)$, (4) the triangle inequality $\delta(x, y) + \delta(y, z) \geqq \delta(x, z)$.

A sequence of points x_n of a metric space is said to "converge" to the limit x, if and only if $\delta(x_n, x) \to 0$ as $n \to \infty$. Convergence evidently has the following three properties: every constant sequence $\{x, x, x, \cdots\}$ converges to x, a sequence converges to at most one limit, and every subsequence of a sequence converging to x converges to x. This fact suggests the abstraction of an L-space as any space in which a notion of "convergence" is defined having these properties.‡

* The German equivalents are Summe, Durchschnitt, Komplement; cf. Hausdorff [1], pp. 14, 17. (Numbers in square brackets refer to the bibliography at the end of the book.) For the notions of a "ring" and "field" of sets, cf. ibid., Chap. V. These definitions will also be found in Alexandroff-Hopf [1].

† First stated in Fréchet's Thesis, *Sur quelques points du calcul fonctionnel*, Rendiconti di Palermo, 22 (1906), 1-74.

‡ The notion is due to Fréchet; cf. Hausdorff [1], p. 230.

By the "closure" of a subset X of a metric space, is meant the set $\bar{X}$ of limits of convergent sequences of points of X. It is easy to prove that

C1: $X \leqq \bar{X}$. C2: $\bar{\bar{X}} = \bar{X}$.

C3*: $\overline{X + Y} = \bar{X} + \bar{Y}$. C4: *If p is a point, then $p = \bar{p}$.*

This suggests the abstraction of a "T_1-space" as any space in which a notion of the "closure" of sets is defined which has these four properties.† A space in which closure has the first three properties is called a "T_0-space," provided $\bar{p} = \bar{q}$ implies $p = q$.

A subset of a T_0-space is called "closed," if and only if it is identical with its closure; it is called "open," if and only if its complement is closed. It follows from C1–C3* that the closed subsets of any T_0-space form a ring of sets, and that the open subsets form another ring.

By a "neighborhood" of a point of a T_0-space, is meant an open set containing the point. In a metric space $x_n \to x$ (in words, the sequence $\{x_n\}$ converges to the limit point x) if and only if every neighborhood of x contains all but a finite number of the x_n. More generally, in any T_1-space, if one defines convergence to have this meaning, one gets an L-space—and in this sense the notion of an L-space is a generalization of the notion of a T_1-space.

So much for topology; we now come to the algebraic concepts which tie up most closely with lattice theory.

It will be assumed that the reader is familiar with some, at least, of the following concepts: group, ring, field, linear space, linear algebra, linear associative algebra, Lie algebra; they will not be defined here.‡

On the other hand, in order to state theorems with the proper degree of generality, we shall want a vocabulary which is much less standardized; the terms which we shall define belong to what may be called general or "universal" algebra.§

By an "abstract algebra," will be meant any system, whose elements can be combined by operations—this definition includes as special cases groups, rings, etc. By a "subalgebra" of an abstract algebra, we mean a subset which includes every algebraic combination of its own elements—this definition includes the usual definitions of subgroup, subring, subfield, subspace, subalgebra, etc., as special cases.

By an "isomorphism" between two algebras admitting the same operations (e.g., two groups or two rings), we mean a one-one element-to-element correspondence which preserves all combinations. By a "homomorphism," is

† Cf. Alexandroff-Hopf [1], pp. 44, 59. The notion of a T_1-space goes back to Riesz (1909); cf. Kuratowski, *Topologie*, Warsaw, 1933, p. 15.

‡ Definitions may be found in Albert [1] and van der Waerden [1]. Albert calls a linear algebra an "algebra" for short, following Dickson; another synonym often found is "hypercomplex algebra."

§ In the sense of A. N. Whitehead, *Universal Algebra*, Cambridge, 1899. The terminology is suggested by van der Waerden [1], and is studied in some detail by G. Birkhoff [6].

meant a many-one correspondence with the same property.* An isomorphism of an algebra with itself is called an "automorphism"; a homomorphism of an algebra with itself (or a subalgebra of itself) is called an "endomorphism."

By a "congruence relation" on an abstract algebra with univalent operations, is meant a division of its elements into subsets called "residue classes," which preserves the univalence of the operations—e.g., with binary operations, makes the subset containing $x \circ y$ depend only on the subset containing x and the subset containing y. By the "quotient-algebra" A mod θ defined by such a congruence relation θ, is meant the algebra whose elements are the residue classes "modulo θ" into which θ divides A, and in which (for example) if X and Y are two residue classes, $X \circ Y$ is defined as the residue class containing all $x \circ y$ $[x \in X, y \in Y]$.

Each homomorphism $A \to B$ between abstract algebras defines a congruence relation on A, whose residue classes are the sets of antecedents of the different elements of B. Moreover B is isomorphic with the quotient algebra defined by this congruence relation from A. Thus there is a many-one correspondence between the congruence relations on A and its homomorphic images.†

The abstract algebras listed above (groups, rings, etc.) have unique one-element subalgebras, contained in all other non-void subalgebras: this subalgebra is the identity in the case of groups, and 0 in the other cases. Moreover every congruence relation on these algebras is determined by the residue class containing this one-element subalgebra—in fact, it is known that in the cases of groups, rings, linear spaces, and linear algebras, the possible such residue classes are the normal subgroups, the ideals, the subspaces, and the invariant subalgebras, respectively.

Again, by the "direct union" $X \times Y$ of two algebras X and Y having the same operations, will be meant the system itself having these operations, whose elements are the couples $[x, y]$ $[x \in X, y \in Y]$, in which algebraic combination is performed component-by-component (e.g., $[x, y] \circ [x', y'] = [x \circ x', y \circ y']$). The direct union of n algebras is defined similarly.

The somewhat complicated general notion of a "free" algebra also applies to lattices. To define this notion, we suppose given a set $\mathfrak{S}$ of postulates, which assert that certain n_i-*ary*‡ operations f_i exist, and satisfy certain identities (like associative or distributive laws).

If A is any algebra satisfying the postulates of $\mathfrak{S}$, it is natural to define a

* E.g., such that if $x \to x'$ and $y \to y'$ under the homomorphism, and $\circ$ is any binary operation, then $x \circ y \to x' \circ y'$. The general definitions of isomorphism, homomorphism, and automorphism can be found in van der Waerden [1], vol. 1, pp. 29, 32.

† These facts are well-known, but it is less generally known that L-spaces can be regarded as a special kind of abstract algebra. This enables one to subsume Kolmogoroff's theory of the continuous images of topological spaces (Alexandroff-Hopf [1], pp. 60–3) under the general remarks just made, and the notion of the Cartesian product of L-spaces (Kuratowski, *Topologie*, p. 135) under the definition of "direct union."

‡ By an "n-ary" operation, is meant one which combines any sequence of n elements to form a single one. Only unary and binary operations will be used in the sequel.

"function" $p(x_1, \cdots, x_n)$ of elements $x_1, \cdots, x_n$ of A, as an element which can be obtained from them after suitable use of the f_i. Thus in a group, $x^{-1}yxx$ would be a "function" of x and y. The set of all "functions" of the x_i is clearly a subalgebra of A, and is the smallest subalgebra containing the x_i; it is called the subalgebra "generated" by the x_i.

It is a theorem of "universal algebra," that there exists for each cardinal number r, an algebra F_r with the three properties: (1) F_r satisfies the given postulates, (2) F_r is generated by r elements x_i, (3) every equality $p(x_1, \cdots, x_r) = q(x_1, \cdots, x_r)$ between functions of the x_i in F_r is a necessary consequence of the identities of $\mathfrak{S}$. This algebra is unique to within isomorphism, and is called "the free algebra with r generators" defined by the set of postulates $\mathfrak{S}$. It has moreover the remarkable property that *every* algebra with r generators which satisfies the postulates of $\mathfrak{S}$ is a homomorphic image of F_r.

CHAPTER I

PARTIALLY ORDERED SYSTEMS

1. Fundamental definition. It is characteristic of mathematics that it employs very few undefined terms. This is especially true of lattice theory, which involves—besides general logical concepts—only the single undefined term "includes." This term is assumed to be characterized by three properties: reflexiveness P1, antisymmetry P2, and transitivity P3, and is probably best introduced through

DEFINITION 1.1: *By a "partially ordered system,"* is meant a system X in which a relation $x \geqq y$ (read "x includes y") is defined, which satisfies*

P1: *For all x, $x \geqq x$.*
P2: *If $x \geqq y$ and $y \geqq x$, then $x = y$.*
P3: *If $x \geqq y$ and $y \geqq z$, then $x \geqq z$.*

We shall use the terms "includes" and "contains" synonymously; we shall also sometimes write $x \leqq y$ to mean $y \geqq x$, and $x > y$ (or $y < x$) to mean that $x \geqq y$ although $x \neq y$. This notation is thoroughly familiar, and standard.

2. Examples. Mathematics abounds with examples of partial orderings. For instance, the subsets of any class I are partially ordered by the relation of set-inclusion. More generally, so are the subsets "distinguished" by any special property; thus the subgroups of any group, the ideals of any ring, the subspaces of any linear space, and in fact the subalgebras of any abstract algebra are partially ordered by the relation of set-inclusion.

The principle just used can be generalized as follows. Let A be any partially ordered system, and X any subset of A. Then the relation $x \geqq y$ is defined between pairs of elements of X, and satisfies P1–P3 there. Thus a relation of inclusion is defined on X by "relativization" from A, and

THEOREM 1.1: *Any subset of a partially ordered system is itself a partially ordered system, relative to the same inclusion relation.*

3. Configurations. A closely related family of partially ordered systems is provided by the notion of a configuration. This is a more or less vague notion; it has been defined in different ways by various authors,† and the most that

* From the German "teilweise geordnete Menge," Hausdorff [1], first ed., Chap. VI, §2. The assumptions go back to C. S. Peirce [1], and were studied by Schröder [1], also. They occur in a fragmentary way in Leibniz' works (circa 1690). Cf. C. I. Lewis, *A Survey of Symbolic Logic*, Berkeley, U. S. A., 1918, pp. 373–87. Thus P1 is Leibniz' Property 7 (p. 380), P2 his Property 17 (p. 382), and P3 his Property 15 (p. 382).

† Cf. E. H. Moore, *Tactical memoranda*, I–III, Am. Jour., 18 (1896), p. 264; A. B. Kempe, Proc. Lond. Math. Soc., 21 (1890), 147–82; F. Levi, *Geometrische Konfigurationen*, Berlin, 1927; also S. Gorn, *On incidence geometry*, Bull. Am. Math. Soc., 46 (1940), 158–67.

can be said with confidence is that the notion of a configuration is included in the notion of a partially ordered system, and includes the notions of a graph, of a combinatorial complex (in the sense of analysis situs), and of a projective geometry.

In a vague way, by a "configuration" is meant a set of geometrical elements (like the vertices, edges, and faces of a polyhedron), related by *incidence*. But incidence can be defined in terms of set-inclusion: two elements of the configuration are "incident," if and only if one contains the other set-theoretically.*

We shall recur to the notion of a combinatorial complex in §18, and to the notion of a projective geometry in Chapter IV; for the present, we shall be content to observe that every significant known example of a configuration, is easily characterized as a partially ordered system.

4. Further examples. The elements of configurations are so often "distinguished" sets (for example, simplexes), that it is interesting to discover partially ordered systems whose elements would not normally be regarded as sets.

The real numbers, ordered with respect to magnitude, form a partially ordered set. Also, the relation of priority partially orders the time continuum of classical physics. In fact, the two are "isomorphic"—that is, there is a one-one correspondence between their elements which preserves the relation of inclusion. Curiously enough, the relativistic notion of "priority" also partially orders events in space-time.

Similarly, one can partially order the real, single-valued functions defined on any domain, by defining $f \geqq g$ to mean that $f(x) \geqq g(x)$ identically. This observation leads to the theory of partially ordered function spaces developed in Chapter VII.

Again, consider the natural integers, and let $m \geqq n$ mean that *m divides n.*

If we associate with each m the division of the integers into the residue classes modulo m, we see this as a special case of the general principle that if X is any family of partitions of a domain (e.g., the Riemann partitions of a line segment), and if we let $\Pi \geqq \Pi'$ mean that Π' is a *refinement* (i.e., subpartition) of Π, then X becomes partially ordered.

And finally, the different topologies imposed on a space (e.g., the "weak" and "strong" topologies of Hilbert space) are partially ordered by a comparison of how tightly (vaguely speaking) they knit the space together.†

5. Elementary lemmas. In §1, the notations $x \geqq y$, $x \leqq y$, $x > y$, and $x < y$ were introduced; we shall now define $x \not> y$ and $x \not< y$ to mean that $x > y$ and $x < y$ are false, respectively. It is easy to prove

LEMMA 1: *If* $x_1 \geqq x_2 \geqq \cdots \geqq x_n \geqq x_1$, *then* $x_1 = \cdots = x_n$.

* This definition of incidence in terms of inclusion occurs in E. H. Moore, op. cit. It justifies our assertion that "the notion of a configuration is included in the notion of a partially ordered system."

† This idea was developed in the author's *On the combination of topologies*, Fund. Math., 26 (1936), 156–66.

Proof: By P3 and induction, $x_i \geqq x_j$ and $x_j \geqq x_i$ for all i, j; hence by P2, $x_i = x_j$, q.e.d. It is a corollary of this "anti-circularity" condition that if $x_1 \geqq \cdots \geqq x_n$ and $x_k > x_{k+1}$ for some k, then $x_1 > x_n$.

LEMMA 2: *For all x, $x \not> x$; also, $x > y$ and $y > z$ imply $x > z$. Conversely, these two conditions imply* P1–P3.

Proof: The two conditions are corollaries of the definition of $x > x$ and of Lemma 1, respectively. Assuming them, define $x \geqq y$ to mean that either $x > y$ or $x = y$. Then P1 is obvious. To prove P2, note that if $x \geqq y$ and $y \geqq x$, then unless $x = y$, $x > y$ and $y > x$, whence $x > x$ by the second condition, contradicting the first. Finally, P3 follows in case $x \neq y$ and $y \neq z$ since $x > y$ and $y > z$ imply $x > z$, and otherwise trivially.

We shall use the conclusions of this section in later proofs, without reference.

6. Omission of P2 yields further examples. Many systems possess ordering relations ρ which satisfy P1 and P3, but not P2. We shall call such system "quasi-ordered."

For example, let R be any ring of integrity with unity, and let $x \rho y$ mean that *x divides y.* Or let Φ be the class of L-spaces, and let $X \rho Y$ mean that X is homomorphic with a subset of Y. Similarly, let Δ be the class of topological linear spaces, and let $X \rho Y$ mean that X is topologically isomorphic with a subset of Y.

Again, let P consist of the real functions having the unit square for domain, and let $f \rho g$ mean that $f(x, y) \geqq g(x, y)$ except on a set of measure zero—or of first (Baire) category. Likewise, let Σ consist of the Borel subsets of the interval [0, 1], and let $S \rho T$ mean that S contains all of T except for a nowhere dense set.

THEOREM 1.2:† *The algorithm of identifying x and y when (and only when) $x \rho y$ and $y \rho x$, yields a partially ordered system from any quasi-ordered system Q.*

Proof: Let $x \sim y$ mean that $x \rho y$ and $y \rho x$. Then (van der Waerden [1], vol. 1, p. 11), $x \sim y$ means that x and y belong to the same subdivision under some partition of Q. Moreover if $x \rho y$, $x \sim x^*$, and $y \sim y^*$, then $x^* \rho x \rho y \rho y^*$, whence $x^* \rho y^*$—that is, the relation is defined consistently over entire subdivisions. The proof of P1 and P3 is now trivial, while P2 follows immediately from our construction of the subdivisions.

The above algorithm specializes in the examples mentioned to the identification of "associate numbers" in the theory of algebraic numbers, to Fréchet's definition of topological dimension, to Banach's definition of linear dimension, and to the usual notion of "equivalent functions."‡

7. Duality. By the "converse" of a relation ρ, is meant the relation γ such that $x \gamma y$ if and only if $y \rho x$ (in words, such that x is in the relation γ to y if

† This result is due to Schröder [1], p. 184. Cf. also H. MacNeille [1] and [2], §2.

‡ Cf. M. Fréchet, *Les dimensions d'un ensemble abstrait*, Math. Ann., 68 (1910), 145–68; S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, Chap. XII; S. Saks, *Théorie de l'Intégrale*, Warsaw, 1933, p. 36.

and only if y is in the relation ρ to x). It is obvious from inspection of conditions P1–P3 that

THEOREM 1.3 (Duality principle): *The converse of any partial ordering is itself a partial ordering.*

DEFINITION 1.2: *By the "dual" X' of a partially ordered system X, is meant that partially ordered system defined by the converse relation on the same elements.*

Since $(X')' = X$, this terminology is legitimate; we shall also often speak loosely of systems isomorphic with X' as dual to X—although it would perhaps be better to speak of them as "anti-isomorphic" to X. It is obvious that partially ordered systems are dual (or anti-isomorphic) in pairs, whenever they are not *self-dual.* Similarly, the definitions and theorems involving partially ordered systems are dual in pairs, when they are not self-dual.*

Many important partially ordered systems are self-dual (i.e., anti-isomorphic with themselves). Thus the system of all subsets of any aggregate is self-dual: the correspondence carrying each subset into its complement is one-one and inverts inclusion. Again, the system of all linear subspaces through the origin of n-dimensional Cartesian space is self-dual: the correspondence carrying each subspace into its orthogonal complement is one-one and inverts inclusion. This is also true of the closed linear subspaces of Hilbert space.

In these cases the self-duality is of period two: the transform $(x')'$ of the transform x' of any element x is x. This is a common trait of self-duality; cf. §20. We shall call self-dualities (or anti-automorphisms) of period two "involutions."

The "duality principle" of Theorem 1.3 extends to algebra, to projective geometry, and to logic, as we shall see later.

8. Minimal and maximal elements. By a "least" element of a subset X of a partially ordered system P, we mean an element $a \in X$ such that $a \leqq x$ for all $x \in X$; by a "minimal" element, we mean one such that $a > x$ for no $x \in X$. By P2 a least element is minimal, but the converse need not be true.

Dual to the notions of least and minimal elements, we may define "greatest" and "maximal" elements; a greatest element is maximal, but the converse need not be true. In fact, by P2, X can have at most one least and one greatest element, whereas it can have many minimal and maximal elements.

THEOREM 1.4: *Any finite subset X of a partially ordered system has minimal and maximal members.*

Proof: Let the elements of X be $x_1, \cdots, x_n$. Set $m_1 = x_1$; select m_k as x_k if $x_k < m_{k-1}$, and otherwise as m_{k-1}. Then m_n will be minimal. Dually, X will have a maximal member.

We shall use the symbols O and I to denote the (unique) least and greatest elements of a partially ordered system P, whenever they exist. Thus in the

* This was Schröder's formulation of the duality principle ([1], vol. 1, p. 315, Thm. 35).

system of all the subsets of any class, I denotes the whole class and O the empty subclass.

We shall say an element x is "between" a and b if and only if $a \geqq x \geqq b$ or $a \leqq x \leqq b$. And a subset X of a partially ordered system P will be called "convex" if and only if it contains with any a and b, every element "between."

9. Simply ordered sets or chains. The relation of inclusion in many partially ordered systems satisfies

P4: *Given x and y, either $x \geqq y$ or $y \geqq x$.*

In other words, of any two elements, one is less and the other greater.

DEFINITION 1.3: *A partially ordered system satisfying* P4 *is said to be "simply ordered," and called a "chain."*

The positive integers, ordered with respect to magnitude, form a chain; so do the real numbers—hence so does time. So many chains can be embedded isomorphically in the (real) time continuum that we may sometimes speak of "first" and "last," instead of "least" and "greatest," elements.

Clearly any subset of a chain is a chain; so is the dual of any chain. Moreover

THEOREM 1.5: *With chains, the notions minimal and least (maximal and greatest) are identical. Hence any finite subset of a chain has a first (= least) and a last element.*

Proof: If $x < a$ for no $x \in X$, then by P4 $x \geqq a$ for all $x \in X$.

Now writing down in order the first element of a finite chain, then the first of the remaining elements, etc., we see that each element is contained in all later elements and (by P2) no others. We conclude

THEOREM 1.6: *Every chain of n elements is isomorphic with the chain of integers $1, \cdots, n$.*

10. Hasse diagrams. In a hierarchy, it is important to know when one man is another's immediate superior. The notion of an immediate superior can be formulated in an abstract partially ordered system, as follows.

DEFINITION 1.4:* *By "a covers b," it is meant that $a > b$, while no x satisfies $a > x > b$ (that is, no x exists between a and b).*

In terms of this definition, it is easy to define the *graph* of any finite partially ordered system X, as the graph whose vertices are the different elements $x, y, z, \cdots$ of X, in which x and y are joined by a segment if and only if x covers y or y covers x. If the graph is so drawn that whenever x covers y, the vertex x is *higher* than the vertex y, it is called a "Hasse diagram"† of X.

* This definition goes back to Dedekind [2], p. 252, who calls a a "next multiple" of b Cf. also G. Birkhoff [1], p. 445. Fr. Klein and O. Ore say "a is prime over b."

† This is a misnomer: the scheme goes back to H. Vogt, *Résolution Algébrique des Équations*, Paris, 1895, p. 91, and probably earlier.

Two examples of Hasse diagrams are drawn in Fig. 1; the first represents the system of all subsets of a class of three elements.

In terms of Definition 1.4, it is also easy to define a "point" or "atom" as an element which covers O. This corresponds to Euclid's definition of a point as "that which has no parts," and of an atom as something "indivisible."†

Hasse diagrams give an easy test for isomorphism of partially ordered systems having few elements: since $x > y$ in a finite partially ordered system if and only

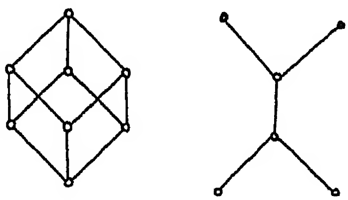

Fig. 1

if there is a chain of elements, each covering the next, connecting x with y, two partially ordered systems are isomorphic if and only if their diagrams can be deformed into each other without letting any segments become horizontal.

Thus they are useful for depicting invariant properties of algebraic systems, and have been applied with this in mind to finite groups. One can also obtain the dual of any partially ordered system just by turning its graph upside down—in other words, by inverting its graph in a horizontal plane.

11. Infinite case. Other applications of the notions of covering and of Hasse diagrams will be given later, but first we shall discuss how the above ideas can be extended to the case of infinite partially ordered systems. Although the definition of the "graph" of a partially ordered system P applies even when P is infinite, that of the "Hasse diagram" of P becomes obscure.

Here the first observation is that a partially ordered system P which is not a chain may nevertheless contain a "relative" chain X. A relative chain X will be called "connected" if and only if it is impossible to interpolate further terms in it: if and only if it contains every $s \epsilon P$ satisfying (i) given $x \epsilon X$, either $x \geqq s$ or $s \geqq x$, and (ii) for some $x, x^* \epsilon X$, $x > s > x^*$.

It is easy to fill in any relative chain of P and so to embed it in a connected relative chain, by interpolating new elements in the chain one at a time, as long as possible. (Of course, this construction involves the axiom of choice.)

A finite chain $x_1 > x_2 > \cdots > x_n$ is connected in P if and only if x_i covers x_{i+1} for all i—that is, if and only if the corresponding sequence of vertices forms a chain in the Hasse diagram of P. But in the infinite case, the graph of a "connected" chain may be graphically disconnected. In fact, no definition of infinite Hasse diagrams is known, having all the properties one would like.

† This special case of Definition 1.4, and its dual, go back at least to Schröder [1], vol. 2, p. 318.

It seems clear, however, that one should regard connected chains as *threads*. Thus consider the familiar past-future diagram of relativistic space-time.* The connected chains in relativistic space-time are the "world-lines"; it is well-known that a line is a possible world-line if and only if the angle it makes with the time-axis is bounded by $\pi/4$. This suggests that in the analytic case one should have some sort of a cone of directions defined at each point, such that a line is a connected chain if and only if it passes through each point in a direction lying within the cone.

12. Dimension. Hasse diagrams suggest a definition of "dimension," in terms of the inclusion relation alone.

By the dimension $d[x]$ of an element x of a partially ordered system P, we mean the maximum "length" d of a chain $x_0 < x_1 < \cdots < x_d = x$ having x for its greatest element. Similarly, by $d[P]$ we mean the maximum length of a chain in P.

It is clear that $d[P]$ is the maximum of the $d[x]$; it is also clear that in determining dimensions one need consider only *connected* relative chains. The notion of dimension is especially important if P has a O and satisfies the

JORDAN-DEDEKIND CHAIN CONDITION: *All connected chains between fixed end-points have the same length.*

Under these circumstances, represent each $x \in P$ by a vertex on a horizontal plane $d[x]$ units above O. If x covers y, then there is a connected chain $x > y > y_1 > \cdots > O$ of length $d[y] + 1$ from x to O, and conversely. Hence x covers y if and only if $x > y$ and $d[x] = d[y] + 1$—and segments in the Hasse diagram of P connect elements on adjacent levels only.

13. Ascending and descending chain conditions. We shall define P to satisfy the "ascending chain condition" (descending chain condition) if and only if *all* its subsets have maximal (minimal) elements.

It is clear by Theorem 1.4 that any finite partially ordered system satisfies both conditions, and even that if $d[P]$ is finite this is true. On the other hand, in many important applications, only one of the conditions holds.†

If P satisfies the descending chain condition, we can argue by induction as follows. If a property is not true of every element of P, then there must be some minimal element m which fails to have it. All $x \leqq m$ will then have this property, from which it follows that to prove that every $m \in P$ has the property, we need only prove it under the added assumption that every $x \leqq m$ has the property.

The familiar notion of a *well-ordered* set fits naturally into this context. A

* Cf. G. D. Birkhoff, *Relativity and Modern Physics*, Harvard University Press, 1927, p. 25. J. M. Keynes, *A Treatise on Probabilities*, London, 1929, p. 39, attempts to handle an analogous problem, but his discussion is quite vague because he does not have the definition of a partially ordered system.

† This fact was proved by Hilbert, for the ideals of many algebraic rings. Cf. van der Waerden [1], vol. 2, pp. 23–7, and §80, from which much of §13 is borrowed.

well-ordered set is (Hausdorff [1], p. 55; the concept is due to Cantor) a simply ordered set which satisfies the descending chain condition (i.e., every subset of which has a first element—which implies P4).

14. Transfinite ordinals. In the present section, we shall extend the constructions of §§8–9 to the "transfinite" case. First consider the proof of Theorem 1.4.

THEOREM 1.7: *A partially ordered system P satisfies the ascending chain condition if and only if it has no infinite ascending sequences—and dually for the descending case.*

Proof: Let X be a subset of P without maximal element. Then we can choose x_1, $x_2 > x_1$, $x_3 > x_2$, $\cdots$ and so on indefinitely, all in X —hence P has an infinite ascending sequence. Conversely, the chain $x_1 < x_2 < x_3 < \cdots$ has no maximal element.

It follows that P satisfies the descending chain condition if and only if all its chains are well-ordered, also, that if P satisfies the descending (ascending) chain condition, then so do all subsets of P.

THEOREM 1.8:† *Let $W \leqq W^*$ mean, with well-ordered sets, that W is isomorphic with a subset of W^*; then* P1 *and* P3 *hold. Moreover applying Theorem* 1.2, *$W \sim W^*$ means that W and W^* are isomorphic, while identification of isomorphic sets yields a well-ordered system. Finally, each W is isomorphic with the set of $W^* \leqq W$ in this system.*

Proof: P1 and P3 are obvious. Now consider the class of all well-ordered sets W^α, and count transfinitely (we are going to extend the proof of Theorem 1.6). Each W^α has a first element x_1, the residual set $W^\alpha - x_1$ has a first x_2, and so on. Proceeding in this way, and dropping out each W^α as soon as all its elements are used up, we see that the W^α are themselves well-ordered with respect to the order of dropping out (of any set of W^α, there is a first which drops out). Moreover W^α dropping out simultaneously are evidently *isomorphic*, while if W^α drops out before W^β, then W^α is isomorphic with a subset of W^β. But conversely, if W^α is isomorphic with a subset of W^β, then no *first* element of W^β, and hence *no* element of W^β, can be counted out earlier than the corresponding element of W^α —whence W^α drops out at least as soon as W^β. Therefore $W^\alpha \leqq W^\beta$ if and only if W^α drops out as soon as or earlier than W^β. The conclusions of Theorem 1.8 are now obvious.

THEOREM 1.9: *A chain is finite if and only if it satisfies both chain conditions.*

Proof: A finite chain evidently satisfies both chain conditions. Conversely, an infinite chain which satisfies the descending chain condition contains by the argument of Theorem 1.8 the infinite ascending sequence $x_1 < x_2 < x_3 < \cdots$; hence no infinite chain satisfies both chain conditions.

† For Theorem 1.8 cf. Hausdorff [1], §13, who attributes the results to Hessenberg. The ideas go back to Cantor and Dedekind.

15. Atomicity. It follows from Theorem 1.6 that a partially ordered system P satisfies the descending chain condition if and only if all its (connected) chains are well-ordered. But it may be that not all the chains of P are well-ordered, and yet that any $a < b$ in P can be joined by at least one well-ordered connected chain.

For example, this is true of the system of all subsets of any continuum—using the axiom of choice, we can add to any set single points one at a time until we exhaust any prescribed overset. It is also true of the system of all closed subsets of any topological space, and of the algebraically closed subfields of any field (since we can adjoin points resp. transcendental elements one at a time).

When this is true, P will be called $\uparrow$-atomic;† and if the dual of P is $\uparrow$-atomic, then P will be called $\downarrow$-atomic. The system of all subsets of any continuum is both $\uparrow$-atomic and $\downarrow$-atomic (being self-dual).

Of course, systems exist which are not atomic at all; viz., the real numbers (ordered by magnitude), Borel sets modulo sets of measure zero (cf. §122; if sets of measure zero are ignored, all connected relative chains are isomorphic with line segments because of the existence of measure), and the system of subsets of the natural integers modulo finite subsets (whose connected relative chains are isomorphic with the chain of points with rational coordinates).

16. A convenient arithmetic notation. One can develop a very concise arithmetic notation involving partially ordered systems, as follows.

The "sum" $X + Y$ of two partially ordered systems X and Y is the system Z whose elements are the elements $x \in X$ plus the elements $y \in Y$. Order is preserved within X and Y, while $x \geqq y$ and $x \leqq y$ are assumed never to hold. Graphically speaking, the Hasse diagram of $X + Y$ is obtained from those of X and Y by laying them beside each other.

The "product" XY of X and Y is the system W whose elements are the couples $[x, y]$ with $x \in X$, $y \in Y$, $[x, y] \geqq [x', y']$ meaning that $x \geqq x'$ and $y \geqq y'$. And finally, the "power" Y^X of Y to exponent X is the system V whose elements are the functions $y = f(x)$ with domain X and range Y for which $x \geqq x'$ implies $f(x) \geqq f(x')$—i.e., the *monotonic* functions $X \to Y$. In Y^X, $f \geqq g$ means that $f(x) \geqq g(x)$ for all $x \in X$.

It has been proved that if the usual notation $1, 2, 3, \cdots, n, \cdots$ is adopted for unordered aggregates (cardinal numbers), then $m + n$, mn, and n^m all keep their meaning, and the usual laws of arithmetic, such as $X(Y + Z) = XY + XZ$, $X^{Y+Z} = X^Y X^Z$, $XXX = X^3$, and so on, continue to hold.‡ Also, if X and X^*, Y and Y^* are dual, then $X + Y$ and $X^* + Y^*$, XY and X^*Y^*, Y^X and Y^{*X^*} are dual.

As a simple instance of the utility of our arithmetic, we may note that if B

† This usage of the word "atomic" goes back in the case of Boolean algebras to A. Tarski [2]; our definition is more elaborate than his. By §116, the two coincide with complete Boolean algebras.

‡ The above definitions were introduced in *An extended arithmetic*, Duke Jour., 3 (1937), 311-6. For that of product cf. the author [1], p. 456—cf. also §22 below.

denotes the chain of two elements, then B^n will denote the system (Boolean algebra) of all subsets of an aggregate of n elements. Also,

THEOREM 1.10: $d[XY] = d[X] + d[Y]$, *and consequently* $d[X_1 \cdots X_n] = d[X_1] + \cdots + d[X_n]$.

Proof: In XY, $[x, y]$ covers $[x', y']$ if and only if $x = x'$ and y covers y', or x covers x' and $y = y'$.

17. Partially ordered sets and T_0-spaces. Let X be any partially ordered system; by the "M-closure" of a subset S of X, we shall mean the set $\bar{S}$ of all t such that $t \leqq s$ for one or more $s \in S$. (We might define the J-closure of S dually.)

By P1, $\bar{S} \geqq S$, and by P3, $\bar{\bar{S}} = \bar{S}$. Again, clearly $\overline{S + T} = \bar{S} + \bar{T}$. And finally, by P2, $\bar{p} = \bar{q}$ implies $p = q$. Hence our definition of M-closure converts X into a T_0-space in the sense of the Foreword.

Moreover in this space, $q \in \bar{p}$ if and only if $q \leqq p$. But conversely, the definition of $q \leqq p$ as meaning $q \in \bar{p}$, partially orders the points of any T_0-space. (Proof: $p \in \bar{p}$; again, $p \in \bar{q}$ and $q \in \bar{p}$ imply $\bar{p} = \bar{q}$ and so $p = q$; finally, $p \in \bar{q}$ and $q \in \bar{r}$ imply $p \in \bar{\bar{r}} = \bar{r}$.)

Now if we consider only finite sets $S = \sum_{i=1}^{n} p_i$, then $\bar{S} = \sum_{i=1}^{n} \bar{p}_i$ is the set of $q \leqq p_i$ for one or more $p_i \in S$, and so the correspondences are reciprocal. We conclude

THEOREM 1.11: *By identifying $q \leqq p$ with the relation $q \in \bar{p}$, we establish a one-one correspondence between finite partially ordered systems X and finite T_0-spaces $\mathfrak{X}$.*

THEOREM 1.12: *In the preceding theorem, B^X is isomorphic with the ring of open subsets of $\mathfrak{X}$, and hence dually isomorphic to the ring of closed subsets of $\mathfrak{X}$.*

Proof: Associate with each subset S of $\mathfrak{X}$ its "characteristic function" f_S: $f_S(p) = 0$ or 1 according as $p \in S$ or $p \notin S$. Then $S \geqq T$ if and only if $f_S \geqq f_T$, and S is closed if and only if $q \leqq p$ implies $f_S(q) \geqq f_S(p)$ —and hence open if and only if f_S is non-decreasing. This establishes the isomorphism asserted.

18. Combinatorial topology. We shall assume the reader to be familiar with the following basic notions of combinatorial topology: simplex, simplicial complex, polyhedron, polyhedral complex—and with their description in terms of cells and matrices of incidence relations.*

It is clear that if we let $x < y$ be a relation between cells meaning "x is on the boundary of y," a complex is just a special kind of partially ordered system. Thus in combinatorial topology, each manifold M is associated with a number of partially ordered systems $P(M)$, any one of which characterizes M to within topological equivalence.

* Cf. S. Lefschetz, *Topology*, American Mathematical Society Colloquium Publications, vol. 12, New York, 1930—Alexandroff-Hopf [1]. The reader is referred to §3 on configurations.

Besides incidence relations, one can define subdivisions, boundaries, cycles, and homology groups directly in terms of $P(M)$. This has been pointed out by Tucker,† who has moreover shown that some of the most rudimentary topological notions apply to any partially ordered system (or "cell space") whatever.

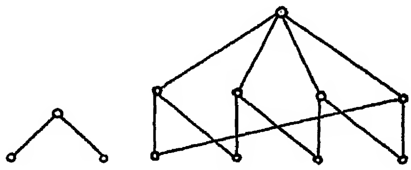

FIG. 2

We may point out in addition that the Jordan-Dedekind chain condition holds in any $P(M)$, while the definition of dimension given in §11 coincides with the usual definition. Furthermore, "products" in the sense of §15 correspond to Cartesian products in the usual sense:‡ $P(M \times M^*) = P(M)P(M^*)$. (To assist the reader, the cell-spaces associated with the line and the square, are graphed in Fig. 2.) Moreover $P(M + M^*) = P(M) + P(M^*)$ and M is connected if and only if the graph of $P(M)$ is connected. Finally, if M' denotes the "dual" of a subdivision of a manifold without boundary, then $P(M')$ is the "dual" of $P(M)$ in the sense§ of §7.

† A. W. Tucker, *An abstract approach to manifolds*, Annals of Math., 34 (1933), 191–243, and esp. *Cell spaces*, ibid., 37 (1936), 92–100. Tucker's theory stands in an interesting relation to our Theorem 1.11.

‡ Cf. E. R. van Kampen, *Die kombinatorische Topologie*, Hague, 1929, p. 17; also Alexandroff-Hopf [1], p. 299.

§ For "dual complexes," cf. Alexandroff-Hopf [1], p. 427, or Veblen, *Analysis Situs*, American Mathematical Society Colloquium Publications, vol. 5_2, New York, 1931, p. 88.

CHAPTER II

LATTICES

19. Definition. The general theory of partially ordered systems is based on a single undefined relation. That of lattices, on the other hand, depends on two dual operations which are definable in terms of this relation, and are analogous in many ways to multiplication and addition.

By an upper bound to a subset X of a partially ordered system P, is meant an element of P containing every $x \in X$. A *least* upper bound is an upper bound contained in every other upper bound (cf. §8). The notions of a lower bound and a greatest lower bound are defined dually—and it is clear by P2 that a set can have at most one l.u.b. (least upper bound) or gr.l.b. (greatest lower bound).

DEFINITION 2.1:* *A lattice is a partially ordered system any two of whose elements x and y have a gr.l.b. or "meet" $x \frown y$, and a l.u.b. or "join" $x \smile y$.*

It is evident that the dual of any lattice is again a lattice, with meets and joins interchanged. Also, it is easy to show by induction that any finite subset of elements of a lattice has a gr.l.b. and a l.u.b.

Again, any minimal element a of a lattice is O, and any maximal element b is I. For unless $a \leqq x$, $a \frown x < x$ —and dually with b. It is a corollary (§14) that any lattice in which the descending chain condition holds has a O, and dually. Hence any finite lattice has a O and a I.

We shall use the words "supremum" and "sup" synonymously with l.u.b.; similarly, we shall use "infimum," "inf," and "common part" synonymously with gr.l.b. Also, "superior" and "inferior" are convenient synonyms for upper bound and lower bound, respectively.

20. Simple examples. A large fraction of the most important partially ordered systems are lattices. Moreover in these systems the operations $\frown$ and $\smile$ usually correspond to familiar and significant constructions.†

Example 1: Let Σ consist of all the subsets of any aggregate I, and let inclusion mean set-inclusion. Then "joins" are set-sums and "meets" are set-products.

* Other terms are: Dualgruppe (Dedekind), Verband (Fr. Klein), structure (Ore). The definition of sums and products in terms of inclusion is due to C. S. Peirce [1], p. 33. Cf. also E. Schröder [1], p. 197, Th. Skolem [1], O. Ore [1]. H. Whitney has suggested the names "cap" and "cup" for the symbols $\frown$ and $\smile$.

† The abundance of lattices in mathematics was apparently not realized before Dedekind [1], pp. 113–4; cf. for example Whitehead [1]. Following Dedekind, Emmy Noether stressed their importance in algebra. Their importance in other domains seems to have been discovered independently by Fr. Klein [1], K. Menger [1], and the author.

Example 2: Let J be the system of positive integers, and let $m \leqq n$ mean "m divides n." Then $m \frown n$ = g.c.f. (m, n) and $m \smile n$ = l.c.m. (m, n).

Example 3: Let S be any set of real numbers, and let $x \leqq y$ have its usual meaning. Then $x \frown y$ is the lesser of x and y, while $x \smile y$ is the greater of x and y. (This remains true if S is any chain—in fact, chains are those partially ordered systems, all of whose subsets are lattices!)

21. Complete lattices. The fact that in any lattice, every finite subset has a gr.l.b. and a l.u.b., suggests asking whether there are many partially ordered systems *all* of whose subsets have a gr.l.b. and a l.u.b. To show that even such "complete" lattices are of common occurrence, we shall want the

DEFINITION 2.2:* *By a "closure operation" on the subsets X of an aggregate I, we shall mean an operation $X \to \bar{X}$ such that*

C1: $\bar{X} \geqq X$,
C2: $\bar{\bar{X}} = \bar{X}$,
C3: $X \geqq Y$ *implies* $\bar{X} \geqq \bar{Y}$.

And by a "closed" set, we shall mean one X identical with its "closure" $\bar{X}$.

THEOREM 2.1: *The subsets "closed" with respect to any closure operation form a complete lattice.*

Proof: The set-product P of any family Φ of closed subsets X_ϕ is closed, since by C3 $\bar{P} \leqq \bar{X}_\phi = X_\phi$ for all $X_\phi \in \Phi$, whence $\bar{P} \leqq P$. Hence P is a gr.l.b. to the X_ϕ. Again, the closure $\bar{S}$ of the set-sum S of the X_ϕ contains every X_ϕ by C1, and is by C3 contained in every closed set containing every X_ϕ. Hence $\bar{S}$ is a l.u.b. to the X_ϕ.

Now note that (1) by C1 I is closed, (2) by the last paragraph, any product of closed sets is closed, and (3) $\bar{X}$ is the least, and therefore the intersection of, the closed sets containing X. E. H. Moore (op. cit) called any property of sets which satisfied (1)–(2) "extensionally attainable"—and showed that conversely the sets possessing any such property were the sets "closed" with respect to (3), understood as a definition of closure.

There is a generalization of Moore's argument which yields a strong presumption that if gr.l.b. *or* l.u.b. exists, then both exist. More precisely,

THEOREM 2.2: *If Γ is a partially ordered system with I, and every subset of Γ has a gr.l.b.—or with O, and all its subsets have l.u.b.—then Γ is a complete lattice.*†

* Cf. E. H. Moore, *Introduction to a Form of General Analysis*, New Haven, 1910, pp. 53–80. I am indebted to Dr. LeRoy Wilcox for this reference. Cf. also the definition of T_0-spaces; evidently C3* implies C3. The distinction between lattices and complete lattices was made by the author [1], p. 442. Cf. M. Ward, *The ideal operators in a lattice*, to appear in Annals of Math.

† It is easy to forget that the existence of I must be assumed; e.g., P. Alexandroff, *Sur les espaces discrets*, Comptes Rendus, 200 (1935), p. 1649.

Proof: Let X be any subset of elements x_α of Γ, and let U be the set of the upper bounds to the x_α. Since U contains I, it is non-void; let a denote the gr.l.b. of U. Since the x_α are all lower bounds to U, a contains every x_α; and since a is a lower bound to U, it is contained in every element containing every x_α. Hence the x_α have the l.u.b. a. The second result (about O and l.u.b.) follows by duality.

22. Examples of complete lattices from algebra. Using the definitions of the Foreword, and Theorem 2.1, we see

THEOREM 2.3: *The subalgebras of any abstract algebra A form a complete lattice.*

Proof: Trivially, A is a subalgebra of itself; also, the intersection of any family of subalgebras is a subalgebra. It is a corollary that the subgroups of any group, the subrings of any ring, the subfields of any field, the subspaces of any linear space, etc., form a complete lattice.

Again, an "operator" on a set I means a transformation of I into itself. Operators on I are indistinguishable from unary operations on I, in the general sense of abstract algebra. Hence we speak of a subset X of I as "closed" with respect to a family Φ of operators on I, if and only if X is transformed within itself by every operator of Φ. And we get from Theorem 2.3

COROLLARY: *The subsets closed with respect to any family of operators form a complete lattice.*

Finally, combining* these two principles, we see that the normal subgroups of any group (i.e., the class of subgroups invariant under all "inner automorphisms"), the characteristic subgroups of any group; the right-ideals, the left-ideals, and the two-sided ideals of any ring; the normal extensions of any field; and the invariant subalgebras of any hypercomplex algebra—all form complete lattices.

These observations pave the way to the study of the structure of abstract algebras.

23. Other examples. Evidently topological closure in any T_0-space is a "closure property" in the sense of §21; hence

THEOREM 2.4: *The closed (and dually, the open) sets of any topological (T_0-)space form a complete lattice.*

Again, referring to §4, where the partially ordered system of the partitions of an abstract class A is considered, it is clear that (1) the *product* of any number of partitions Π_α of A, formed by taking the intersections of the parts into which the different Π_α divide A, is a gr.l.b. to the Π_α, and (2) the degenerate partition into just one part acts as I. Hence by Theorem 2.2,

* We are here using repeatedly the fact that any logical product of closure properties is itself a closure property: the sets "closed" with respect to all the members of any family of "closures" include I and any product of "closed" sets.

THEOREM 2.5: *The partitions of any aggregate form a complete lattice.*

In case A is an abstract algebra, the above remarks also apply to those partitions which are "congruence relations," and show that the congruence relations on any abstract algebra form a complete lattice. The latter is well-known to be isomorphic in the cases of groups, rings, and hypercomplex algebras to the lattices of normal subgroups, ideals and invariant subalgebras discussed in the Foreword on algebra, respectively.*

There are many other examples of complete lattices. Among them may be mentioned: convex bodies in space (the property of convexity is extensionally attainable), the subharmonic functions having a given set of continuous values on the boundary of the unit circle (the harmonic function with these boundary values is a I, and any l.u.b. of subharmonic functions is subharmonic), etc., etc. Cf. for example Chapter VII.†

Finally, any dual, product or power of complete lattices is itself complete; we shall omit the proof.

24. Lattices as abstract algebras. There is a profound analogy between the expressions $x \smile y$, $x \frown (y \smile z)$, $\cdots$ met with in lattice theory and expressions like $x + y$, $x(y + z)$, $\cdots$ familiar from elementary algebra.‡ For instance, one readily verifies the following as consequences of Definition 2.1,

L1: $x \frown x = x$ *and* $x \smile x = x$.
L2: $x \frown y = y \frown x$ *and* $x \smile y = y \smile x$.
L3: $x \frown (y \frown z) = (x \frown y) \frown z$ *and* $x \smile (y \smile z) = (x \smile y) \smile z$.
L4: $x \frown (x \smile y) = x$ *and* $x \smile (x \frown y) = x$.

called respectively the idempotent, commutative, associative and absorptive laws (the names are mostly Boole's).

THEOREM 2.6: *Identities* L1–L4 *completely characterize lattices.*

Proof: In any lattice, $x \geqq y$ if and only if $x \frown y = y$ (or dually, $x \smile y = x$). But in *any* system satisfying L1–L4, $x \frown y = y$ if and only if $x \smile y = x \smile (x \frown y) = x$; moreover if one defines $x \geqq y$ to mean $x \frown y = y$, then one gets a *lattice* in which $x \frown y$ and $x \smile y$ are the greatest lower and least upper

* Actually, the lattice of congruence relations is a sublattice of the lattice of partitions (G. Birkhoff [6], Thm. 24). But we shall not use this fact in the sequel.

† It seems to the author that study of the lattice of all partitions of n given objects (which is highly analogous to the symmetric group of all permutations of the objects) would prove highly fruitful in combinatory analysis—which greatly needs systematization. The nature of this lattice has a direct bearing on Kirkman's Schoolgirls Problem and related questions (hence indirectly on finite plane projective geometries). Cf. P. Dubreil and M.-L. Dubreil-Jacotin, *Théorie algébrique des relations d'équivalence*, Jour. de Math., 18 (1939), 63–96.

‡ This was first pointed out by George Boole, *The Mathematical Analysis of Logic*, Cambridge, 1847. Boole, however, did not know the postulational method. L1 and L2 on addition occur in Leibniz, who also in a sense anticipated Theorem 2.7 (cf. C. I. Lewis, *A Survey of Symbolic Logic*, Berkeley, 1918, p. 376, Thm. 6, and p. 383, Prop. 21).

bounds of x and y, respectively.* For example, $x \frown x = x$ implies P1. Again, if $x \geqq y$ and $y \geqq x$, then $y = x \frown y = y \frown x = x$ by L2 and hypothesis, proving P2. While if $x \geqq y$ and $y \geqq z$, then by L3, $x \frown z = x \frown (y \frown z) = (x \frown y) \frown z = y \frown z = z$, whence $x \geqq z$. Finally, since by L1–L3 $x \frown (x \frown y) = (x \frown x) \frown y = x \frown y$, $x \frown y$ is a lower bound to x; by L2, it is therefore a lower bound to y also. But it is a *greatest* lower bound since $x \geqq z$ and $y \geqq z$ imply $(x \frown y) \frown z = x \frown (y \frown z) = x \frown z = z$ by L3. Dually, $x \smile y$ is the least upper bound of x and y, completing the proof.

Remark: It follows from the proof that any binary operation $x \circ y$ which is idempotent, commutative and associative defines, through the convention that $x \geqq y$ means $x \circ y = y$, a partially ordered system in which $x \circ y =$ gr.l.b. (x, y).†

25. Generalized commutative and associative laws. It is well-known from algebra, that any binary operation $x \circ y$ which is commutative and associative, defines an operation $x_1 \circ \cdots \circ x_n = x_1 \circ (x_2 \circ \cdots \circ x_n)$ which combines any finite sequence $x_1, \cdots, x_n$ of addends, by a composition invariant under permutations of the addends—and conversely. If besides $x \circ x = x$, then repeated occurrences of an addend are equivalent to a single occurrence: $x_1 \circ \cdots \circ x_n$ depends only on the *set* of the x_k. Moreover if $S_1, \cdots, S_r$ are any sets of elements, and $f(S_i)$ denotes the result of combining the elements in S_i, then $f(S_1) \circ \cdots \circ f(S_r) = f(S)$, where S is the set-union of the S_i. Moreover the laws we have just stated have a meaning even with infinite sets whereas iterations of a binary operation can never combine more than finite addends. This leads us to state the following (transfinite) *generalized* laws,

L*1: *Any set S of elements has a* meet $f(S)$ *and a* join $g(S)$, *depending only on S.*

L*2: *If Φ is any family of sets S_ϕ, and S denotes the set-union of the S_ϕ, then $f(S)$ is the meet of the $f(S_\phi)$ and $g(S)$ is the join of the $g(S_\phi)$.*

These, combined with L4, characterize complete lattices.‡

The following useful generalization of the $\sum$-$\prod$-notation of analysis is due to Peirce [2]. If the members of a subset S of a lattice are denoted x_i, then one often denotes sup S by $V_S x_i$ and inf S by $\Lambda_S x_i$.

26. Definitions suggested by general algebra. The analogy between the characterization of lattices by L1–L4 and the usual definitions of groups, rings, etc., suggests the application of the general terminology of abstract algebra to lattices.

* As Dedekind [1], p. 109 has pointed out, L1 can be proved from L4: $x = x \frown [x \smile (x \frown y)] = x \frown x$, and dually $x = x \smile x$. The separation of L1–L4 from the other identities of Boolean algebra was first accomplished by E. Schröder [1].

† This was remarked by Huntington [1], p. 294. One may also note here that $O \frown x = O$, $O \smile x = x$, $x \frown I = x$, $x \smile I = I$ for all x. The first three of these identities are analogous to the familiar laws $0x = 0$, $0 + x = x$, and $x1 = x$ of arithmetic.

‡ For these laws, cf. G. Birkhoff [1], p. 442. They are less interesting than the generalized distributive law L6 which will be discussed in Chapter V, but are of a piece with it.

Thus they lead us to define a "sublattice" of a lattice L as a subset which contains with any two elements their join and their meet. The reader should be warned of the anomaly whereby a subset of L may be a lattice with respect to the inclusion relation in L, although not a sublattice of L. Thus the subgroups of any group form a lattice with respect to set-inclusion, yet (except in the case of cyclic groups) they are not a sublattice of the lattice of all subsets.†

An isomorphism between lattices is by §4 simply a one-one correspondence which preserves inclusion (and hence joins and meets); an automorphism is an isomorphism of a lattice with itself. But the situation becomes ambiguous when we try to define *homomorphisms* between L and a second lattice L^*.

There may exist many-one correspondences $L \to L^*$ which preserve inclusion but neither joins nor meets, ones which preserve meets but not joins, and ones which preserve both.‡ We shall term these order-homomorphisms, join-homomorphisms, meet-homomorphisms, and lattice-homomorphisms respectively. Clearly the antecedents of any element under a lattice-homomorphism form a convex sublattice.

This situation is evidently going to be similar when we try to discuss congruence relations on a lattice—again, there is a quadruple ambiguity. Moreover the analogy with groups and rings breaks down in another respect. Although a congruence relation on any group (or ring) is specified by the normal subgroup (resp. ideal) of elements congruent to its identity (resp. zero), congruence relations on chains and many other lattices§ need not be specified by the "ideal" of elements congruent to their O.

27. Polynomials and identical relations. The analogy mentioned at the beginning of §26 leads one to define a "lattice polynomial," as any polynomial function of variables $x_1, \cdots, x_n$, which is either a *monomial* x_i, or a *join* or *meet* of polynomials already defined. Lattice polynomials have a number of non-trivial properties, which we shall now consider.‖

THEOREM 2.7: *Let* $f(x_1, \cdots, x_n)$ *be any lattice polynomial, and suppose* $x_i \leqq y_i$ *for all* i. *Then* $f(x_1, \cdots, x_n) \leqq f(y_1, \cdots, y_n)$: *lattice polynomials are* monotonic *functions.*

† Similarly, by a σ-sublattice of a σ-lattice, we mean a subset which contains all joins and meets of countable aggregates of its members—and by a complete sublattice of a complete lattice, one which contains *all* joins and meets of its members.

We note without proof that the subalgebras of an abstract algebra A which are carried within themselves by any set of endomorphisms of A form a sublattice of the lattice of all subalgebras of A. (Thus the normal and characteristic subgroups of any group form sublattices of the lattice of all its subgroups.)

‡ These distinctions were first made explicitly by O. Ore [1], p. 416. Thus the correspondence between the subsets of an abstract algebra and the subalgebras they generate is a *join*-endomorphism which does not preserve meets.

§ Boolean algebras and complemented modular lattices, however, behave like groups in this respect—cf. Theorem 4.7.

‖ The one-sided distributive law is due to Schröder [1], p. 280; the one-sided modular law to Dedekind [1], p. 120.

Proof: By induction, it suffices to show that $x \leqq x'$ implies $x \frown y \leqq x' \frown y$ and $x \smile y \leqq x' \smile y$. But if $x \leqq x'$, then $x \frown y = (x \frown x') \frown y = x \frown (x' \frown y) \leqq x' \frown y$ and similarly $x' \smile y = (x' \smile x) \smile y = x' \smile (x \smile y) \geqq x \smile y$.

Corollary 1: *We have the one-sided distributive laws* $x \frown (y \smile z) \geqq (x \frown y) \smile (x \frown z)$ *and* $x \smile (y \frown z) \leqq (x \smile y) \frown (x \smile z)$.

Proof: By Theorem 2.7, $x \frown (y \smile z)$ is an upper bound to both $x \frown y$ and $x \frown z$; hence it contains their join. The second law follows by duality.

Assuming $x \geqq z$ in Corollary 1, we get the self-dual one-sided modular law,

Corollary 2: *If* $x \geqq z$, *then* $x \frown (y \smile z) \geqq (x \frown y) \smile z$.

Corollary 3: *Always* $(x \frown y) \smile (u \frown v) \leqq (x \smile u) \frown (y \smile v)$.

Proof: By Theorem 2.7, $x \frown y \leqq (x \smile u) \frown (y \smile v)$; similarly, $u \frown v \leqq (x \smile u) \frown (y \smile v)$. The conclusion is now obvious.

28. Free lattices. The general notion of a "free algebra" (cf. the Foreword) also applies to lattices.*

A lattice L is said to be "generated" by elements $a_1, \cdots, a_n$, if and only if every element is a polynomial function of the a_i; clearly every subset of L generates a sublattice of L.

By the "free lattice with generators $x_1, \cdots x_n$," is meant the set of formal lattice polynomials $f(x_1, \cdots, x_n)$, $g(x_1, \cdots, x_n)$, etc., where (1) two polynomials are considered to be the same element if and only if they determine identical functions in every lattice, and (2) polynomials are combined formally. For example, $(x \frown y) \frown x$ and $(x \frown y)$ are considered to be the same element, and the join of $(x \frown y)$ and y is defined as $(x \frown y) \smile y$.

In virtue of Theorem 2.6, the system F_n so defined *is* a lattice generated by the x_i; moreover it is *universal* in the sense that if L is any lattice generated by n elements $a_1, \cdots, a_n$, then there is a lattice-homomorphism $F_n \to L$ under which $x_1 \to a_1, \cdots, x_n \to a_n$.

The free lattice generated by two symbols x and y has four elements: x, y, $x \frown y = O$, and $x \smile y = I$. On the other hand, the free lattice generated by three symbols can be shown by an example (by a lattice of partitions) to be infinite.†

A very important problem of lattice theory is that of finding an "Entscheidungsverfahren" for lattice polynomials: a rule telling in a finite number of steps whether or not two given polynomials are equal. This problem has recently been solved by P. M. Whitman (*The free lattice with n generators*, to appear in the Annals of Math.).

29. The combination of lattices. In the terminology of §16, the "sum" of

* For more details concerning this notion, cf. the author [6], p. 440—where general applications are also given.

† Cf. the author [6], p. 452.

two lattices is never a lattice: for if x and y come from different summands, they have no upper bound.

THEOREM 2.8: *The "product" of two lattices is the direct union of the lattices, regarded as abstract algebras. Hence it is always a lattice.*

Proof: In the notation of §16, $[x \smile x', y \smile y']$ is not only an upper bound to $[x, y]$ and $[x', y']$, but it is contained in every other upper bound—hence it is a least upper bound. The proof is completed by duality.

It is a corollary that the product of two lattices is a σ-lattice (respectively complete) if and only if both factors are σ-lattices (respectively complete).

Furthermore, if X is any lattice and Y is any partially ordered system (of n elements), then X^Y is a lattice—it is in fact a sublattice of X^n.

30. Unicity theorem on product decompositions. Clearly an element x of a product P of partially ordered systems X_i is a I for P, if and only if each X_i-component of x is a I for X_i—and dually for O. Hence P has a O and I if and only if every X_i does. In the present sections we shall consider only this case.

We shall then define a_i as the element of P whose X_i-component is I and whose other components are O. Then the $x \leqq a_i$ will form a subset X_i^* of P isomorphic with X_i. Moreover for any $x \in P$, the X_i-component of $x \frown a_i$ is the same as that of x, and its others are O—whence $x \frown a_i \in X_i^*$ and $x = \sup_i (x \frown a_i)$.

Again, if x is fixed, the $t \leqq x$ are the elements each of whose components is contained in the corresponding component of x. Hence

LEMMA: *The set T of $t \leqq x$ is the product of factors isomorphic with the sets T_i^* of $t_i \leqq x \frown a_i$.*

Now suppose P can be factored into X_i and also into Y_j. Define elements a_i as above, and analogously elements b_j such that the $x \leqq b_j$ form a set Y_j^* isomorphic with Y_j. Finally, denote by Z_j^i the set of $t \leqq a_i \frown b_j$. Then by the lemma, each X_i^* is the product of the Z_j^i with superscript i, and each Y_j^* that of the Z_j^i with subscript j. We infer the

THEOREM 2.9: *Associated with any two factorizations of a partially ordered system P with O and I, into factors X_i and Y_j respectively, is a factorization into Z_j^i, such that the product of the Z_j^i with fixed i is X_i, and the product of the Z_j^i with fixed j is Y_j.*

COROLLARY 1: *The factorization of P into indecomposable factors is* unique, *in the strict sense that any factorization of P is obtainable by grouping these factors into subfamilies.*†

31. The center of a lattice. The ideas of the preceding section suggest the notion of the "center" of a partially ordered system with O and I. By this we

† These results were first proved by the author for distributive lattices in [1], p. 457, and then for general lattices in [3], p. 616. Throughout, one-element factors are ignored: otherwise there would be no indecomposable systems.

mean‡ the set of elements $a \in P$ which have one X_i-component I and the others O under some representation of P as a product of X_i.

By the commutativity and associativity of products, we see that for this it is necessary and sufficient that $a = [I, O]$ under some representation $P = X_i Y$ of P as the product of two factors. And from this condition we can read off

LEMMA 1: *The center of any partially ordered system is preserved under dual automorphisms. Each element of the center has a unique complement, also in the center.*

Proof: Dualization carries $[I, O]$ into $[O, I]$, which is in the center. Again, $[I, O] \frown [x, y] = [x, O]$ and $[I, O] \smile [x, y] = [I, y]$; hence $[x, y]$ is a complement of $[I, O]$ if and only if $x = O$ and $y = I$.

Now note that the proof of the unicity of product decompositions brings out the fact that, if a_i and b_j are in the center, then so is $a_i \frown b_j$. By duality (cf. Lemma 1) then so is $a_i \smile b_j$, and we can conclude

THEOREM 2.10: *The center of any lattice is a sublattice preserved under dual automorphisms.*

As an example of the usefulness of the notion of the "center" of a lattice, we may note that the center of the lattice of all subgroups of a finite group G contains those and only those subgroups occurring as factors in representations of G as the direct product of (normal) subgroups, of relatively prime order. Hence if G is Abelian or "hypercentral," they are the Sylow-subgroups and their products.

32. Polarity; complementation. Let ρ be any dyadic relation defined between the members of any two abstract aggregates I and I^*. We shall write $x \rho x^*$ if x *is* in the relation ρ to x^* $[x \in I, x^* \in I^*]$, and $x \bar{\rho} x^*$ if x is *not* in the relation ρ to x^*. We do not assume I and I^* to be different.

If X is any subset of I, denote by X^* the set of x^* such that $x \rho x^*$ for all $x \in X$; reciprocally, if Y is any subset of I^*, denote by $Y\dagger$ the set of $x \in I$ such that $x \rho y$ for all $y \in Y$. Clearly

LEMMA 1: *If $X \geqq X_1$ in I, then $X^* \leqq X_1^*$ in I^*; similarly, if $Y \geqq Y_1$ in I^*, then $Y\dagger \leqq Y_1$ in I.*

LEMMA 2: *For any subset X of I, $((X^*)\dagger)^* = X^*$; and for any subset Y of I^*, $((Y\dagger)^*)\dagger = Y\dagger$.*

Proof of Lemma 2: By hypothesis, $x \rho x^*$ for all $x \in X$, $x^* \in X^*$; hence $X \leqq (X^*)\dagger$; similarly, $Y \leqq (Y\dagger)^*$ for any subset Y of I^*. But setting $Y = X^*$ in the last inequality, we get $X^* \leqq ((X^*)\dagger)^*$, while by Lemma 1 $X \leqq (X^*)\dagger$ implies $X^* \geqq ((X^*)\dagger)^*$. Combining, $X^* = ((X^*)\dagger)^*$; similarly, $Y\dagger = ((Y\dagger)^*)\dagger$.

‡ The definition is generalized from the notion of "center" of a complemented modular lattice as defined by J. von Neumann [2], Part I, pp. 38-40, and Part III, p. 1.

THEOREM 2.11: *The operations* $X \to (X^*)\dagger$ *and* $Y \to (Y\dagger)^*$ *are* closure *operations; moreover the correspondences* $X \to X^*$ *and* $Y \to Y\dagger$ *are (reciprocal)* dual isomorphisms *between the lattices of "closed" subsets of* I *and* I^*.

Proof: Clearly $(X^*)\dagger \geqq X$; using Lemma 1 twice, $X \geqq X_1$ implies $(X^*)\dagger \geqq ((X_1)^*)\dagger$; using Lemma 2, $(((X^*)\dagger)^*)\dagger = (X^*)\dagger$—hence $X \to (X^*)\dagger$ is a closure operation. Similarly, the correspondence $Y \to (Y\dagger)^*$ defines a closure operation. Now by Lemma 2, the X^* and $Y\dagger$ are closed, and the correspondences $X^* \to (X^*)\dagger$ and $Y\dagger \to (Y\dagger)^*$ are reciprocal—hence they are one-one. And by Lemma 1, they invert inclusion, which completes the proof.

Example 1: Let I be any finite Abelian group, let I^* be the group of its characters, and let $x \rho \chi$ $[x \in I, \chi \in I^*]$ mean $\chi(x) = 0$. Then the closed subsets of I and I^* are their subgroups—and the correspondence of Theorem 2.11 defines a dual isomorphism between the subgroup-lattice of I and that of I^*. (Actually, I and I^* are isomorphic!)

Example 2: Let $I = I^*$ be any ring, and let $x \rho y$ mean that $xy = O$. Then every X^* is a right-ideal, every $X\dagger$ a left-ideal. In the semi-simple case, the converse is also true, and the correspondence of Theorem 2.11 establishes a dual isomorphism between the lattice of left-ideals and that of right-ideals, which carries two-sided ideals into two-sided ideals.

Again, we might let I be a field, and I^* a finite group of automorphisms α of I. If we define $x \rho \alpha$ to mean $\alpha(x) = x$, we get the typical situation of Galois theory.‡ Or we might let I be any Banach space, and I^* the conjugate space of linear functionals $\lambda(x)$ on I. If we define $x \rho \lambda$ to mean $\lambda(x) = 0$, we discover that the "closed" subsets of I and I^* are their closed subspaces—whence the lattices of these are dually isomorphic.

COROLLARY 1: *If* $I = I^*$ *and the relation* ρ *is* symmetric *(i.e., if* $x \rho y$ *implies* $y \rho x$*), then* $X\dagger = X^*$, *so that we have just* one *closure operation* $X \to \bar{X} = (X\dagger)^* = (X^*)\dagger$. *Moreover the correspondence* $X \to X^*$ *is an "involution" of the lattice of "closed" subsets.*

In this case, we shall call $X^* = X\dagger$ the "polar" of X, and write it X'. We will then have

K1: $(X')' = X$.
K2: $(X \frown Y)' = X' \smile Y'$ *and* $(X \smile Y)' = X' \frown Y'$.

Example 3: Let I be any group, and let $x \rho y$ mean that $xy = yx$ (x commutes with y). Then the $\bar{X}$ are all subgroups (but not conversely), and the "involution" carries any $\bar{X}$ into its "centralizer."

Example 4: Let I be any class, and let $x \rho y$ mean $x \neq y$. Then *every* set is "closed," and the involution carries every set into its set-complement.

Example 5: Let I be Cartesian n-space, and let $x \rho y$ mean $x \perp y$ (x is or-

‡This case was discussed in part by M. Krasner, Jour. de Math., 17 (1938), 266–86.

thogonal to y). Then the "closed" subsets are the linear subspaces, and the involution carries every subspace into its orthogonal complement.‡

Example 6: Let I be a vector lattice (Chapter VII), and let $x \rho y$ mean that $|x| \frown |y| = O$. Then the "closed" subsets are the complemented normal subspaces.

COROLLARY 2: *If the relation is* anti-reflexive (*if $x \rho y$ implies $x \neq y$*), *then X and X' are* complementary, *in the sense that*

L7: $X \frown X' = O$ *and* $X \smile X' = I$.

Proof: Since $X \frown X'$ contains only x such that $x \rho x$, $X \frown X' = O$. But I is closed, and so the involution must interchange O with I. By K2, also $X \frown X'$ with $(X \frown X')' = X' \smile (X')' = X' \smile X = X \smile X'$, whence $X \smile X' = I$. We note that the hypotheses of Corollary 2 are fulfilled in Examples 4, 5, and 6.

33. Normal subsets. Consider the construction of §32, when ρ is the inclusion relation of a partially ordered system P. Then by definition, X^* is the set of lower bounds, and $X\dagger$ the set of upper bounds to X. We shall call $\bar{X} = (X\dagger)^*$ the "normal hull" of X, and shall call X a "normal subset" if and only if it is its own normal hull.

It is evident that if x is any element of P, then $x\dagger$ is the set of $t \geqq x$, and by P3 $(x\dagger)^*$ is the set of x^* of $t \leqq x$. That is, $\bar{x}$ consists of the subelements (M-closure) of x.

THEOREM 2.12: *Any partially ordered system has a one-one representation by sets which preserves inclusion and meets.*

Proof: Represent each element by its normal hull. By P1 and P2, distinct elements have distinct normal hulls; hence the representation is one-one. If $x \leqq y$, then $x^* \leqq y^*$ by P3 and conversely; hence the representation preserves inclusion. Finally, u is a lower bound of a set X of elements x_α, if and only if $u^* \leqq X^*$; while u contains every lower bound if and only if $X^* \leqq u^*$. Thus (as J. von Neumann has remarked) $u = \inf X$ if and only if $u^* = X^*$; dually, $u = \sup X$ if and only if $u\dagger = X\dagger$. But X^* is by definition the set-product of the x_α^*; hence the representation carries infima into set-products, completing the proof.

We shall see later that in general no one-one representation exists which carries infima into set-products *and* suprema into set-unions; this would imply the distributive law, which need not be true.

COROLLARY 1: *The inclusion relation is characterized completely by postulates* P1–P3.

Incidentally, if in any complete lattice L we regard combination into joins as one algebraic operation, and each operator $x \to x \frown a$ $[a \in L]$ as a unary operation, we convert the sets of Theorem 2.12 into its "subalgebras." Hence

‡ Evidently any symmetric bilinear form defines a polarity.

by stretching our definitions, we can infer that any complete lattice is the lattice of all subalgebras of a suitable "abstract algebra" (cf. the author [1], Thm. 5.1).

34. Completion by cuts. We shall now give MacNeille's generalization ([1] and [2], §11) of Dedekind's celebrated construction of irrational numbers by "cuts" (*Stetigkeit und irrationale Zahlen*, Brunswick, 1892, p. 11).

THEOREM 2.13: *Any partially ordered system P can be embedded in a complete lattice, with preservation of inclusion, greatest lower bounds, and least upper bounds.*

Proof: Consider the normal subsets of P; by Theorem 2.1, they form a complete lattice. By Theorem 2.12, the correspondence between elements of P and their normal hulls embeds P in this lattice in a way preserving inclusion and gr.l.b. It remains to show that l.u.b. are preserved.

To see this, form the *dual* P' of P. Then P' can be (dually) embedded in the complete lattice of $(X^*)\dagger$ in P, with preservation of gr.l.b.—and hence so that l.u.b. in P correspond to gr.l.b. of $(X^*)\dagger$. But the correspondence $\bar{X} \to \bar{X}\dagger$ is one-one between normal subsets of P and the $(X^*)\dagger$, whence gr.l.b. of $(X^*)\dagger$ correspond to l.u.b. of normal subsets, completing the proof.

Caution: Although MacNeille's construction yields $[-\infty, +\infty]$ from the rational numbers, there are other ways of embedding partially ordered systems, and especially lattices, in complete lattices, besides his.‡ We shall discuss them in Chapter III, §54.

35. Intrinsic topology of chains. The next three sections will be devoted to defining *intrinsic topologies* in lattices—intrinsic in the sense of being definable in terms of the inclusion relation alone.

First the case of simple ordering is discussed; this case is relatively easy and essentially known.‖ Then (§36) a general notion of convergence is defined, closely tied up with generalized notions of lim inf and lim sup. Finally (§37) this convergence is altered by a construction due to Urysohn to yield a topology tied up with metric notions.

In a chain, define an "open interval" either by fixing a "lower end-point" a, and forming the set of $t > a$, or by fixing an "upper end-point" b, and forming the set of $t < b$, or by fixing both, and forming the set of $a < t < b$. Clearly the intersection of any finite number of open intervals is itself a (possibly void) open interval—having for its lower and upper end-points the greatest resp. least corresponding end-points of the intersecting intervals.

Then define a "neighborhood" of any x as any open interval containing x.

‡ In particular, when applied to algebraic integers, partially ordered with respect to divisibility, MacNeille's construction does not yield the lattice of ideals necessarily. For example, polynomials in n variables over any field form a Gaussian ring; hence when partially ordered with respect to divisibility, they already form a complete lattice—yet not all ideals are principal ideals.

‖ The author has been unable to find a precise reference; cf. F. Hausdorff, *Grundzüge einer Theorie der geordneten Mengen*, Math. Ann., 65 (1908), 435–55.

THEOREM 2.14: *Any chain is a normal Hausdorff space relative to its intrinsic topology—and the latter is invariant under dualization.*

This means that postulates (1)–(8) of Hausdorff [1], pp. 228-9 are satisfied by the neighborhood system (the proofs, which are simple, are left to the reader)—and that dualization carries neighborhoods into neighborhoods, which is quite obvious. Incidentally, a normal Hausdorff space is a T_1-space.

In this space, open sets are the sums of open intervals, and closed sets are the complements of open sets—that is, the residual sets left after open intervals have been removed. Moreover dense sets are those having elements on every interval.

But in any Hausdorff space, $x_n \to x$ means that every neighborhood of x contains all but a finite number of terms of the sequence $\{x_n\}$.

THEOREM 2.15: *For $x_n \to x$, it is necessary and sufficient that sequences $\{u_n\}$ and $\{v_n\}$ exist, satisfying $u_n \leqq u_{n+1} \leqq x_{n+1} \leqq v_{n+1} \leqq v_n$ and* $\sup \{u_n\} = \inf \{v_n\} = x$.

That is, it is necessary and sufficient that the x_n be contained in a sequence of nested closed intervals intersecting in x.

Proof: Suppose $x_n \to x$, and define u_n as the *least* element of the set x, x_n, $x_{n+1}, \cdots$. Either $x = u_n$, or $x_k < x$ for some $k \geqq n$, and (by the hypothesis of convergence), only a finite number of x_i fail to exceed x_k. The least of these will be u_n, which thus exists in any case. But now by definition, $u_n \leqq u_{n+1} \leqq x_{n+1}$. Also, x is an upper bound to the u_n, while $y < x$ implies that almost every u_n exceeds y—whence x is the *least* upper bound of the u_n. Similarly, a dual sequence $\{v_n\}$ exists.

Conversely, if two such sequences exist, and $a < x < b$, then some u_m must exceed a, some v_n be inferior to b, whence all but the first $(m + n)$ terms of $\{x_k\}$ must be in the given neighborhood of x.

36. On (o)-convergence in general. It is natural to ask what happens to the considerations of §35 in the case of *partially* ordered systems.

One surprising thing happens: the definition of an "open interval" has no obvious extension. Indeed, consider the plane, letting $(x, y) \leqq (x', y')$ mean that $x \leqq x'$ and $y \leqq y'$. Then $(-1, 0) < (0, 0) < (1, 0)$, and yet the set $(-1, 0) < (x, y) < (1, 0)$ should not be considered a neighborhood of $(0, 0)$—at least, unless we wish to get a (trivial) totally discrete topology from the definition.

On the other hand, the conditions of Theorem 2.15 extend very nicely. We shall therefore make

DEFINITION 2.3: *By $x_n \to x$, we mean that sequences $\{u_n\}$ and $\{v_n\}$ exist, such that* (1) *for all n, $u_n \leqq x_n \leqq v_n$*, (2) *$\{u_n\}$ is monotone increasing, and $\{v_n\}$ monotone decreasing, and* (3) $\sup \{u_n\} = \inf \{v_n\} = x$. *Under these circumstances, we shall say the x_n "(o)-converge" to x, and summarize conditions* (2)–(3) *by writing $u_n \uparrow x$ and $v_n \downarrow x$.*

THEOREM 2.16: *Relative to (o)-convergence, any partially ordered system is an L-space in the sense of Fréchet, and any product $X_1 \cdots X_n$ of such systems is topologically the Cartesian product of its factors.**

Proof: The proofs of Fréchet's postulates are immediate; it is even obvious that convergence is not altered by any permutation of the terms of a sequence. The truth of the second assertion is evident since betweenness, monotonicity, and suprema and infima are all reckoned component-by-component in a product of partially ordered systems (cf. §29).

Now let us restrict our attention to lattices in which every bounded countable set has a supremum and an infimum.† Such systems we shall call "conditional σ-lattices"—and if *all* countable subsets have suprema and infima, we shall just call them σ-lattices. A lattice in which *every* bounded set has a supremum and an infimum will be called "conditionally complete."‡

In a conditional σ-lattice, we shall define

$$\limsup \{x_n\} = \inf_n \{\sup_{k \geqq n} x_k\},$$

$$\liminf \{x_n\} = \sup_n \{\inf_{k \geqq n} x_k\}.$$

Clearly these exist for any (bounded) sequence, and we always have lim sup $\geqq$ lim inf.

THEOREM 2.17: *The sequence $\{x_n\}$ (o)-converges to x if and only if* lim sup $\{x_n\}$ = lim inf $\{x_n\} = x$.

Proof: If $x_n \to x$, then $u_n \leqq \inf_{k \geqq n} x_k \leqq \sup_{k \geqq n} x_k \leqq v_n$, whence lim inf $\{x_n\} \geqq \sup_n u_n = x$ and dually, which shows that certainly lim inf $\{x_n\} \geqq$ lim sup $\{x_n\}$. Conversely, if lim inf equals lim sup, we can set $u_n = \inf_{k \geqq n} x_k$, $v_n = \sup_{k \geqq n} x_k$, and show that $x_n \to x$ in the sense of (o)-convergence.

It is a corollary that (o)-convergence specializes to Hausdorff's definition of the convergence of a sequence of sets ([1], p. 19); we shall see in Chapter VII that under a wide variety of circumstances it specializes to E. H. Moore's "relative uniform convergence" of functions.

37. Star-convergence. It seems to be a basic principle of general topology

* For L-spaces, cf. the Foreword; for Cartesian products, cf. Hausdorff [1], p. 102—convergence of a sequence means simply convergence of the various components individually. A definition of (o)-convergence applicable to σ-lattices was introduced by the author [6] (cf. Thm. 29), and independently by Kantorovitch [1]. Cf. Hausdorff [1], p. 19.

† By a "bounded" subset of any partially ordered system, we mean one having an upper and a lower bound. In order that every (bounded) countable set have a supremum and an infimum, it is sufficient that this be true for monotone sequences. Note the close analogy between the definition of a σ-lattice and Hausdorff's definitions of a σ-ring and σ-field of sets ([1], §18).

‡ We note that in order to complete a "conditionally complete" lattice by cuts, one need only adjoin an O and an I.

that most spaces can be "topologized" by superficial considerations—but that such a topology has no value except in virtue of its general properties.

If we apply this principle to (o)-convergence, we are troubled by several things. We may have $x_n \to x$ and $y_n \to y$ without having $x_n \frown y_n \to x \frown y$ or $x_n \smile y_n \to x \smile y$: that is, the lattice operations need not be continuous in the (o)-topology.† However, for this it is sufficient that $y_n \uparrow y$ imply $x \frown y_n \uparrow x \frown y$ and dually: continuity in each variable implies simultaneous continuity in the two variables. (Proof: If $u_n \uparrow x$, $u_n^* \uparrow y$, $v_n \downarrow x$, $v_n^* \downarrow y$, $u_n \leqq x_n \leqq v_n$, and if $u_n^* \leqq y_n \leqq v_n^*$, then $u_n \frown u_n^* \uparrow$, $\sup u_n \frown u_n^* \geqq \sup_n u_m \frown u_n^* = u_m \frown y$ (by the hypothesis of continuity in each variable) for all m; we infer $\sup u_n \frown u_n^* \geqq \sup u_m \frown y = x \frown y$ using this hypothesis again. But the reverse inequality is obvious; hence $u_n \frown u_n^* \uparrow x \frown y$—and dually $v_n \frown v_n^* \downarrow x \frown y$, whence $x_n \frown y_n \to x \frown y$ by definition. The proof is completed by duality.) Note that by the generalized associative law in any case $x_n \downarrow x$ and $y_n \downarrow y$ imply $x_n \frown y_n \downarrow x \frown y$—and dually.

Another trouble is that derived sets may fail to be closed. Also, we may have $x_j^i \to x_i$ for all i, and $x_i \to x$, without having any diagonal sequence $x_{j(i)}^i \to x$. And finally, the assumption that every subsequence of $\{x_n\}$ contains a subsubsequence (o)-converging to x need not imply $x_n \to x$. Each of these three situations is pathological, in the sense of being impossible for metric convergence in a metric space.

The last situation can be eliminated. For in any L-space, by making the definition: "x_n star-converges to x, if and only if every subsequence of $\{x_n\}$ contains a subsubsequence converging to x," one gets‡ a new L-space without the last deficiency, in which "derived sets" (closures) are the same as in the original topology. Star-convergence is remarkable in that it gives the metric topology in metric lattices.

In closing, we note that if the lattice operations are continuous in the (o)-topology, then they are continuous in the star topology. For any subsequences $\{x_{n(k)}\}$ and $\{y_{n(k)}\}$ of sequences star-converging to x resp. y, will contain subsubsequences (o)-converging to x resp. y and the corresponding subsubsequences $\{x_{n(k(i))} \frown y_{n(k(i))}\}$ will (o)-converge to $x \frown y$ by hypothesis, whence by definition $\{x_n \frown y_n\}$ star-converges to $x \frown y$. The proof is completed by duality.

38. Directed sets and generalized limits. In this section, we shall study a generalized notion of "convergence to a limit," based on a special order-property, and having many applications to topology and analysis.

† This was shown by the author [6], p. 453. We shall define a "topological lattice" as one without this deficiency; the definition is analogous to Schreier's definition of a topological group (Abh. Hamb., 4 (1926), 15–32). The lattice of all subgroups of the direct sum of countable cyclic groups of prime order is not topological. Although $x_n \uparrow x$ implies $x_n \frown a \uparrow x \frown a$, still $x_n \downarrow x$ fails to imply $x_n \smile a \downarrow x \smile a$.

‡ Cf. P. Urysohn, *Sur les classes (L) de M. Fréchet*, Enseignement Math., 25 (1926), 77–83. Star-convergence in lattices was introduced independently by Kantorovitch ([1], p. 143), and von Neumann and the author (Annals of Math., 38 (1937), p. 56).

DEFINITION 2.4: *A "directed set"*† *is a partially ordered system any two of whose elements have an upper bound (or "common successor"). A directed set* $\{x_\alpha\}$ *of points of a topological space* Σ *"converges" to a limit* x, *if and only if to every neighborhood* $U(x)$ *of* x *corresponds a member of* $\{x_\alpha\}$, *all of whose successors lie in* $U(x)$.

Thus any lattice is a directed set, and in particular ordinary and transfinite sequences (being chains) are directed sets. A partial ordering which yields a directed set is also said "to have the property of Moore-Smith."

THEOREM 2.18: *A directed set converges to at most one limit, in any Hausdorff space.*‡

This important *unicity* property follows since if $x_\alpha \to x$ and $x_\alpha \to y$, then any neighborhoods $U(x)$ of x and $V(y)$ of y would determine elements x_α and x_β, such that $U(x)$ contained all successors of x_α and $V(y)$ all those of x_β. Hence $U(x)$ and $V(y)$ would both contain every common successor of x_α and x_β, and so be overlapping—whence by our last hypothesis $x = y$.

Since the set-product of two open sets is open (C3*), the neighborhoods $U(x)$ of any point x of any topological space Σ are a directed set under the convention that the successors of any $U(x)$ are the neighborhoods of x contained in $U(x)$. It follows that the closure of any subset S of Σ is the set $\bar{S}$ of the limits of the convergent directed sets of points of S. This permits one to correlate in a consistent fashion general topological ideas flowing out of the intuitive notion of *convergence* with those flowing out of the dual notions of *closure* and *neighborhood* (closed and open sets). This is not otherwise possible in general function-spaces.

Again, the Riemann partitions of any line interval into subintervals Δ_i are a directed set§ under the convention that the successors of any partition Π are its refinements. Hence if $f(p)$ is any function with domain $[0, 1]$ and range lying in a topological linear space X, the Riemann integral of $f(p)$ as usually defined is the (unique) "limit" in the sense of Definition 2.3, of the finite sums $\sum m(\Delta_i)f(p_i)$ $[p_i \in \Delta_i]$.

† This notion is due to E. H. Moore, *Definition of limit in general integral analysis*, Proc. Nat. Acad. Sci., 1 (1915), p. 628. The ideas of the present section are stated in full detail in the Annals of Math., 38 (1937), 39-56. There it is shown also that one can "complete" (in the sense of van Dantzig) topological algebras without assuming countability axioms. The generalized construction used for this includes Prüfer's construction for completing algebraic systems. And finally, any completed topological linear space has the property that its closed totally bounded sets are compact—although the converse is false. This justifies von Neumann's definition of such a topological linear space as "complete," and correlates it with van Dantzig's.

‡ Or H-space in the sense of Alexandroff-Hopf [1]—i.e., any T_0-space in which any two distinct points have disjoint neighborhoods. Hausdorff spaces are T_1-spaces (Theorem).

§ Actually, they are a distributive lattice, isomorphic with the distributive lattice of the sets of points of subdivision. Likewise, the Lebesgue partitions described in the next paragraph form a lattice (which is not usually distributive), and the finite subsets considered two paragraphs below form a generalized Boolean algebra in the sense of Stone.

Similarly, if the domain of $f(p)$ is any space with a σ-ring of measurable sets, then its partitions into countable measurable subsets S_i of measures $m(S_i)$ form a directed set, and so one can define the Lebesgue integral of $f(p)$ as the (unique) "limit" of the unconditional sums $\sum m(S_i)f(p_i)$ $[p_i \in S_i]$.

And indeed, the definition of an "unconditional sum" is itself most conveniently stated in terms of directed sets. For the finite subsets S of any aggregate I are a directed set when ordered by the relation of set-inclusion. Hence if I consists of elements x_γ of a topological linear space X, one can let the statement "x is the unconditional sum of the x_γ" mean that the finite partial sums converge to x in the sense of Definition 2.3.

We can easily define generalized convergence with respect to order in any partially ordered system: by $x_\alpha \to x$, we mean that a monotone decreasing set $\{u_\alpha\}$ and a monotone increasing set $\{v_\alpha\}$ exist, such that $u_\alpha \geqq x_\alpha \geqq v_\alpha$ for all α while $\inf \{u_\alpha\} = \sup \{v_\alpha\} = x$. This definition makes every normal subset closed; it also makes every lattice topologically dense in the complete lattice of its cuts. In a complete lattice, we can also define

$$\limsup x_\alpha = \inf_\beta \, (\sup_{\alpha \geqq \beta} x_\alpha),$$

$$\liminf x_\alpha = \sup_\beta \, (\inf_{\alpha \geqq \beta} x_\alpha)$$

with the assurance that $\limsup \{x_\alpha\} = \liminf \{x_\alpha\}$, the equality holding if and only if the x_α converge.*

39. Methods of enumeration. Let L be any finite partially ordered system with O. Various authors† have defined a "Möbius function" μ on L, as follows: $\mu[O] = 1$, and $\mu[x] = -\sum_{y<x} \mu[y]$. The reason for calling μ a "Möbius function" is that, if L is the lattice of positive integers ordered by the relation "x divides y," then $\mu[x]$ becomes the Möbius function of number theory.

* Unsolved problems: It is not known which sets are closed (i.e., contain all their limit points) or open (i.e., have closed complements). Neither is it known when the topology of a lattice is obtainable by topological relativization from the complete lattice of its cuts.

† The Möbius function of number theory was defined in the early nineteenth century. The computation of the number of ways of coloring a map in λ colors was first undertaken by G. D. Birkhoff (Proc. Edin. Math. Soc., ser. 2, 2 (1930), also Annali di Pisa, ser. 2, 3 (1934)), and later by H. Whitney (*A logical expansion in mathematics*, Bull. Am. Math. Soc., 38 (1932), 572–9). The application of the above ideas of enumeration to subgroups of p-groups is due to P. Hall (Proc. Lond. Math. Soc., 36 (1933), 29–95).

L. Weisner (Trans. Am. Math. Soc., 38 (1935), 474–84) first, and P. Hall (Quar. Jour., 7 (1936), 134–51) independently gave the method an abstract lattice-theoretic formulation. Hall (op. cit.) also enumerated the number of ways of generating certain groups by λ elements, and showed that (1) if $\lambda(x; n)$ denotes the number of chains of length n which can be interpolated between O and x, then (the proof is by induction) $-\mu[x] = \lambda(x; 1) - \lambda(x; 2) + \cdots$, (2) if L is a lattice, then $\mu[x] = O$ unless x is the join of elements covering O, and (3) the Möbius function of the dual of L is that of L. Cf. also M. Ward, *The algebra of lattice functions*, Duke Jour., 5 (1939), 357–71.

There is also a connection between the Möbius function of a combinatorial complex and its Euler-Poincaré characteristic.

If in addition L satisfies the Jordan-Dedekind chain condition, and $d[x]$ denotes the dimensions of x, then we can associate with every $x \in L$ a "characteristic polynomial," defined through the relation

$$p_x[\lambda] = \lambda^{d[x]+1} - \sum_{y<x} p_y[\lambda]. \tag{39.1}$$

This is connected with the Möbius function by the fact that if $\mu_y[x]$ $[y \leqq x]$ denotes the Möbius function for x relative to the subsystem of elements containing y, then $p_x[\lambda]$ equals $\sum_{y \leqq x} \lambda^{d[y]+1} \mu_y[x]$. (Thus the coefficient of λ in $p_x[\lambda]$ is $\mu[x]$.)

40. Application to four color problem. Now consider the celebrated problem of coloring any map on the plane in four colors. If, by a "submap" of a map Ω, we mean a map formed from Ω by obliterating boundaries, then the submaps of any Ω form a lattice with O which satisfies the Jordan-Dedekind chain condition (the dimension of any submap is one less than the number of its regions). Moreover $p_\Omega[\lambda]$ is the number of ways of coloring Ω in λ colors so that no two adjacent regions are in the same color: the proof rests in (39.1) and the fact that there are λ^n ways of coloring n regions in λ colors, each of which colors either Ω or a submap of Ω so that no two adjacent regions are in the same color.

Similarly, the principle that any choice of λ elements of a finite group G generates either G or a subgroup S of G leads to a function expressing the number of ways of generating G by a given number of elements—the function being a polynomial determined by the lattice of the subgroups of G. And finally, a like enumeration principle (cf. P. Hall, Proc. Lond. Math. Soc., 36 (1933), 29–95) yields theorems on the number of subgroups of different orders p^k in "p-groups" of prime-power orders p^n. (The fact that the subgroup-lattice of any p-group satisfies the Jordan-Dedekind chain condition is used there.)

CHAPTER III

MODULAR LATTICES

41. Definition. Just as the elements of many (but not all!) groups satisfy the identity $xy = yx$, so the elements of many (but not all!) important lattices satisfy the following important "modular identity" (or mixed associative law),*

L5: *If* $x \leqq z$, *then* $x \smile (y \frown z) = (x \smile y) \frown z$.

Definition 3.1: *A lattice will be called "modular," if and only if its elements satisfy the modular identity.*

Theorem 3.1: *A lattice is non-modular if and only if it contains the lattice of Fig. 3 as a sublattice.*

Proof: The lattice graphed fails to satisfy L5. Conversely, unless L5 holds in a lattice, by the one-sided modular law we have

$$u = x \smile (y \frown z) < (x \smile y) \frown z = v.$$

But in this case the five elements u, v, y, $u \smile y$, and $v \frown y$ form a sublattice isomorphic with the one of Fig. 3. For

$$u \smile y \leqq v \smile y \leqq (x \smile y) \smile y = x \smile y = x \smile (y \frown z) \smile y = u \smile y,$$

whence $v \smile y = u \smile y$. Dually, $u \frown y = v \frown y$—and now it is clear by L4 that we have a sublattice.

Corollary 1: *In a modular lattice, $v > u$ is incompatible with $y \smile u = y \smile v$ and $y \frown u = y \frown v$; moreover this condition conversely implies modularity.*

Corollary 2: *A lattice which is non-modular contains a sublattice not satisfying the Jordan-Dedekind chain condition.*

Corollary 3: *In a modular lattice, (ξ') if x and y cover a, and $x \neq y$, then $x \smile y$ covers x and y, and dually (ξ'') if a covers x and y, and $x \neq y$, then x and y cover $x \frown y$.*

(Proof: Unless $x \smile y$ covered (say) x, the sublattice generated by x, y, and any element between x and $x \smile y$ would be isomorphic with the lattice graphed in Fig. 3.)

42. Examples of modular lattices. The importance of modular lattices is deeply rooted in

* The modular identity was discovered by Dedekind [1], p. 115. Dedekind also proved Theorem 3.1 and Corollaries 2-3 (cf. [2], IX, X, XIV). Corollary 1 is due to G. Bergmann [1]. Other terms for modular lattice are "Dedekindscher Verband" (Fr. Klein), and "Dedekind structure" (O. Ore).

THEOREM 3.2: *The normal subgroups of any group form a modular lattice.**

Proof: In any lattice (§30), we have the one-sided modular law. Now suppose X, Y, and Z are normal subgroups of a group, and that $X \leqq Z$. Then $X \smile Y$ certainly contains all products $xy\ [x \in X,\ y \in Y]$; while conversely, since $(xy)(x'y') = (xx')(x'^{-1}yx'y')$ and $(xy)^{-1} = y^{-1}x^{-1} = (y^{-1}xy)^{-1}y^{-1}$, the xy form a group containing X and Y—whence $X \smile Y$ *is* the set of the xy. Conse-

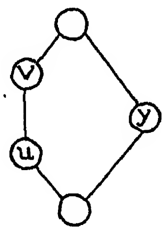

FIG. 3

quently if $u \in (X \smile Y) \frown Z$, then $u = xy = z\ [z \in Z]$, and $y = x^{-1}z \in Z$ (since $x \in X \leqq Z$), which implies $u = xy \in X \smile (Y \frown Z)$. We conclude $X \smile (Y \frown Z) \geqq (X \smile Y) \frown Z$, which with P2 and the one-sided modular law implies L5.

It is a corollary that the subgroups of any commutative group form a modular lattice, since they are all normal.

Again, since L5 is self-dual, the dual of any modular lattice is a modular lattice. Obviously also any sublattice of a modular lattice, and any lattice-homomorphic image of a modular lattice, is modular. Moreover any product $L_1 \cdots L_n$ of modular lattices is modular: if, in $L_1 \cdots L_n$, $[x_1, \cdots, x_n] \leqq [z_1, \cdots, z_n]$, then by definition $x_k \leqq z_k$ for all k, and so $x_k \smile (y_k \frown z_k) = (x_k \smile y_k) \frown z_k$ for any set of $y_k \in L_k$, from which we conclude $x \smile (y \frown z) = (x \smile y) \frown z$ in $L_1 \cdots L_n$, completing the proof. It is a corollary, since if X is any partially ordered system, L^X is a sublattice of some product $L \cdots L$, that any *power* L^X of a modular lattice L is modular.

43. Groups with operators. One can greatly generalize Theorem 3.2, by using Krull's notion of a "group with operators."† By an "operator" on a group G is meant simply an endomorphism $\omega: x \to x\omega$ of G; by a "group with operators" is meant a group G and a fixed set Ω of endomorphisms of ("operators on") G. By an "Ω-subgroup" of a group with operators is meant a subgroup which contains with any x every $x\omega\ [\omega \in \Omega]$.

The concept of a "group with operators" helps a great deal to unify modern algebra. If Ω consists of the inner automorphisms of G, the Ω-subgroups are its

* The proof given here is due to Dedekind [2], p. 270, 1900. Cf. also Dirichlet's *Zahlentheorie*, §169, pp. 498–9.

† W. Krull, *Über verallgemeinerte endliche Abelsche Gruppen*, Math. Zeits., 23 (1925), 161–86, and *Theorie und Anwendung der verallgemeinerten endlichen Abelschen Gruppen*, Heidelberg Sitz. (1926). Cf. also van der Waerden [1], vol. 1, p. 132.

normal subgroups; if Ω consists of all automorphisms of G, they are its characteristic subgroups; if Ω consists of all endomorphisms, the Ω-subgroups are the "strongly characteristic" subgroups of Chatelet.

Again, if G is the additive group of any ring R, and Ω consists of the right-multiplications $x \rightarrow xb$, then the Ω-subgroups are the right-ideals of R; if Ω consists of the left-multiplications $x \rightarrow ax$, they are its left-ideals; if Ω consists of all multiplications $x \rightarrow axb$, they are its two-sided ideals.

Finally, if G is the additive group of any linear space or linear algebra A, and Ω consists of the scalar multiplications $x \rightarrow x\lambda$, then the Ω-subgroups are the linear subspaces of A. If Ω contains also the left- and right-multiplications $x \rightarrow ax$ and $x \rightarrow xb$, the Ω-subgroups become the invariant subalgebras of A. If Ω contains only the left-multiplications, the Ω-subgroups become the left-invariant subalgebras; similarly with right-multiplications and right-invariant subalgebras. While if Ω is a representation of any group or ring, the Ω-subspaces are the subspaces "half-reducing" the representation in representation theory.

LEMMA: *The Ω-subgroups of any group with operators are a sublattice of the lattice of all subgroups.*

The proof involves only the concepts "abstract algebra" and "endomorphism," construed in the most general sense. Indeed, let A be any abstract algebra, let Ω be any family of endomorphisms of A, and let X and Y be any subalgebras of A carried into themselves by every endomorphism of Ω (we may call them "Ω-subalgebras"). Then each $\omega \in \Omega$ carries elements in both X and Y into elements in both X and Y; hence the meet (set-product) $X \frown Y$ is an Ω-subalgebra. Likewise, it carries any algebraic combination of elements of X and Y into a similar algebraic combination; hence it carries the set $X \smile Y$ of all such combinations into itself, and $X \smile Y$ is an Ω-subalgebra.

44. Other modular lattices. It is a corollary of Theorem 3.2 and the preceding lemma that the normal subgroups of any group, the characteristic subgroups of any group, the subspaces of any linear space, the right-ideals, left-ideals, and ideals of any ring, the right-invariant, left-invariant and invariant subalgebras of any linear algebra,* and the subspaces which "half-reduce" any representation of a group or ring—all these form modular lattices.

We shall see later (§78) that any abstract projective geometry is a modular lattice. Further, all the examples of distributive lattices and Boolean algebras listed in Chapters V and VI are a fortiori modular.

Also, the normal subfields of any algebraically closed fields form a modular lattice, in virtue of the dual automorphism between the lattice of normal sub-

* These results were probably first stated by the author ([1], Thms. 26.1, 27.2, and §28). A claim can be made that they were known by Dedekind; it seems quite sure that they were known to Emmy Noether (cf. Dedekind [2], p. 271). It is perhaps of historical interest that they were almost surely unknown to Wedderburn when he wrote his classic paper on semi-simple linear associative algebras (*On hypercomplex numbers*, Proc. Lond. Math. Soc., 6 (1908), 77–118).

fields and the lattice of normal subgroups of the group of automorphisms of the field (cf. E. Steinitz, *Algebraische Theorie der Körper*, p. 143).

45. Definitions involving quotients. In the theory of modular lattices L, a central rôle is played by the notion of "quotient," as defined in the following.*

DEFINITION 3.2: *A "quotient" of L is a symbol x/y, where x and y are in L and $x \geqq y$. A quotient x/y is called "prime" if and only if x covers (is prime over) y. Two quotients of the form $v/u \frown v$ and $u \smile v/u$ are called "transposes," while two quotients x/y and x'/y' are called "projective" (in symbols, $x/y \sim x'/y'$) if and only if there exists a sequence x/y, x_1/y_1, x_2/y_2, $\cdots$, x'/y' in which any two successive quotients are transposes.*†

Clearly transposition is reflexive and symmetric, while projectivity is reflexive, symmetric and transitive. Hence (van der Waerden [1], vol. 1, p. 14) the quotients of any modular lattice are divided into a number of disjoint classes, such that any two quotients in the same class, and no two quotients in different classes, are projective. Also, the definitions are self-dual.

46. Dedekind's transposition principle. We shall now prove a transposition principle due to Dedekind ([2], XI, p. 259; cf. also Ore [1], p. 418), namely

THEOREM 3.3: *Let L be any modular lattice, and let u and v be any two elements of L. Then the correspondences $x \to u \smile x$ and $y \to v \frown y$ are reciprocal isomorphisms between the set X of elements x between $u \frown v$ and v, and the set Y of elements y between u and $u \smile v$. Moreover they carry quotients in X resp. Y into transposed quotients.*‡

Proof: If $u \frown v \leqq x \leqq v$, then $u = u \smile (u \frown v) \leqq u \smile x \leqq u \smile v$, and if $x \leqq x'$, then $u \smile x \leqq u \smile x'$; hence the first correspondence is monotonic (and single-valued) from X to a subset of Y. Dually, the second correspondence is monotonic from Y to a subset of X. But $x \in X$ implies $v \frown (u \smile x) = (v \frown u) \smile x = x$ by L5 and the definition of X; dually, $u \smile (v \frown y) = y$ for all $y \in Y$. Thus the two correspondences are reciprocal, hence both one-one, and so isomorphisms. To see that they yield transpositions, note that they carry x/x' into $u \smile x/u \smile x'$, where $u \smile x = x \smile (u \smile x')$ and, since $x \geqq x'$, $x' = (x \frown u) \smile x' = x \frown (u \smile x')$.

COROLLARY 1: *Any quotient projective to a prime quotient is itself prime.*

COROLLARY 2: *If u and v both cover w, then $u \smile v$ covers both u and v. Dually, if w covers both u and v, then both u and v cover $u \frown v$.*

* The notion is due to Dedekind [2], p. 245, where it is however given a numerical connotation. Cf. also the author [1], p. 452, and more especially O. Ore [1], [2].

† The term "transposable" is due to Ore; the term "projective" is due to von Neumann. The idea of transposition goes back to Dedekind; that of projectivity is due to Ore (who speaks of "similar" quotients). Ore used the notion in studying non-commutative polynomials—whose theory is thus correlated with projective geometry. Ore also defined $x/y \geqq x'/y'$ to mean $x \geqq x'$ and $y \geqq y'$, thus constructing L^B (cf. §16).

‡ We note that the sets X and Y are convex sublattices of L.

Incidentally, Theorem 3.1 shows that the condition of Theorem 3.3 is sufficient as well as necessary for modularity.

THEOREM 3.4: *The lattice generated by the sublattices X of elements between $u \frown v$ and u, and Z of elements between $u \frown v$ and v, is the product XZ.*

Proof: Denote $x \smile z$ by $[x, z]$; then by L2–L3,

$$[x, z] \smile [x', z'] = (x \smile z) \smile (x' \smile z') = (x \smile x') \smile (z \smile z') = [x \smile x', z \smile z'].$$

But now by Theorem 3.3 respectively L5, we have

$$[x, z] = x \smile z = [(v \smile x) \frown u] \smile z = (v \smile x) \frown (u \smile z).$$

Hence $[x, z] \frown [x', z'] = [(v \smile x) \frown (v \smile x')] \frown [(u \smile z) \frown (u \smile z')]$ by L2–L3. And by Theorem 3.3, $(v \smile x) \frown (v \smile x') = v \smile (x \frown x')$ and $(u \smile z) \frown (u \smile z') = u \smile (z \frown z')$, whence

$$[x \frown x', z \frown z'] = [v \smile (x \frown x')] \frown [u \smile (z \frown z')] = [x, z] \frown [x', z'],$$

completing the proof. We note that u and v are in the center of XZ.

COROLLARY 1: *If $(x_1 \smile \cdots \smile x_k) \frown x_{k+1} = a$ for $k = 1, \cdots, n$, then the lattice generated by the sublattices X_k of elements between a and x_k is $X_1X_2 \cdots X_n$ —and conversely.*

(The converse is obvious.) But the definition of $X_1X_2 \cdots X_n$ is symmetric in the subscripts—it is not changed by any permutation of them. Hence one can legitimately state*

DEFINITION 3.3: *Under the hypotheses of the preceding corollary, the x_k are said to be "independent" over a.*

The notion of independence over O contains as special cases the notions of "disjointness" in set theory, and of linear independence of subspaces of a linear space. The notion of "independence under a" can clearly be defined dually.

47. Lattice homomorphisms. Let θ be any congruence relation on a lattice of finite dimensions. We shall say that θ *annuls* a quotient x/y if and only if $x \equiv y\ (\theta)$.

LEMMA: *The relation θ is determined by the set of prime quotients which it annuls.*

Proof: It is sufficient to show that $u \equiv v\ (\theta)$ if and only if θ annuls every prime quotient between $u \frown v$ and $u \smile v$. But if $u \equiv v\ (\theta)$, then $u \frown v \equiv u \frown u = u \smile u \equiv u \smile v\ (\theta)$, and so θ annuls every quotient between $u \frown v$ and $u \smile v$, while conversely, we can connect any u and v via $u \smile v$, by a sequence whose successive terms differ by a prime quotient between $u \frown v$ and $u \smile v$; hence if θ annuls every such prime quotient, successive terms, and so all terms, are congruent mod θ.

* J. von Neumann [2], vol. 1, p. 11; Theorem 3.4 is nearly the same as von Neumann's Theorem 1.2. Cf. also Fr. Klein, Deutsche Math., 2 (1937), 216–41.

But now in a *modular* lattice, $x \smile y \equiv x\ (\theta)$ implies $y = y \frown (x \smile y) \equiv y \frown x\ (\theta)$; and dually, $x \frown y \equiv y\ (\theta)$ implies $x = x \smile (x \frown y) \equiv x \smile y\ (\theta)$. Hence if θ annuls a quotient, it annuls all transposed quotients—and therefore all projective quotients. Consequently,

THEOREM 3.5: *A congruence relation on a modular lattice of finite dimensions is determined by which sets of projective prime quotients it annuls.*

48. Jordan-Dedekind chain condition. We shall now prove a result due ultimately to Jordan, and in its abstract form to Dedekind,*

THEOREM 3.6: *Suppose L is of finite dimensions, and let p be any prime quotient in L. Then the number $d[p, x, \gamma]$ of occurrences of quotients projective to p in any connected chain γ from 0 to x is a function $d[p, x]$ independent of γ.*

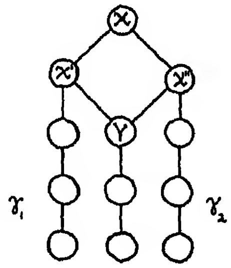

FIG. 4.

Proof: Define $d[p, y/z]$ as 1 or 0 according as y/z is or is not projective to p. Since L is of finite dimensions, if Theorem 3.5 is false $d[p, x, \gamma_1] \neq d[p, x, \gamma_2]$ for some minimal x. We shall show this is impossible. Indeed, if x covers the same element x' in γ_1 and in γ_2, then since $x' < x$,

$$d[p, x, \gamma_1] = d[p, x'] + d[p, x/x'] = d[p, x, \gamma_2].$$

While if x covers x' in γ_1 and $x'' \neq x'$ in γ_2, then since x is minimal, $d[p, x']$ and $d[p, x'']$ are single-valued. Since besides the pairs of quotients $x/x' = (x' \smile x'')/x'$ and $x''/(x' \frown x'')$, and x/x'' and $x'/(x' \frown x'')$ are transposes, we have (denoting $x' \frown x''$ by y)

$$\begin{aligned} d[p, x, \gamma_1] &= d[p, x'] + d[p, x/x'] = d[p, y] + d[p, x'/y] + d[p, x/x'] \\ &= d[p, y] + d[p, x''/y] + d[p, x/x''] \\ &= d[p, x''] + d[p, x/x''] = d[p, x, \gamma_2], \end{aligned}$$

completing the proof (cf. also Fig. 4).

Since the length $d[x, \gamma]$ of any connected chain from O to x is $\sum_p d[p, x, \gamma]$, we get as a corollary

* C. Jordan, *Traité des Substitutions*, Paris, 1870, p. 663. Dedekind [2], p. 264.

Theorem 3.7: *The Jordan-Dedekind chain condition holds in any modular lattice of finite dimensions.*

49. Covering conditions for modularity. The best tests for modularity in a finite lattice are the covering conditions (ξ')–(ξ'') of Corollary 3 to Theorem 3.1. We shall show later that they are sufficient* as well as necessary. But first,

Theorem 3.8: *Each of conditions (ξ') and (ξ'') implies the Jordan-Dedekind chain condition, in a lattice of finite dimensions.*

Proof: We can substitute (ξ'') for Dedekind's transposition principle in the proof of Theorem 3.6; this is clear if we use Fig. 4. If (ξ') is assumed, one can argue dually.

Theorem 3.7 is one corollary of this. It is another corollary (§12) that the dimension function $d[x]$ makes x covers y imply $d[x] = d[y] + 1$.

Lemma: *The inequality $d[x] + d[y] \geqq d[x \cup y] + d[x \cap y]$ is implied by (ξ'); the reverse inequality is implied by (ξ'').*

Proof: Introduce the locution "x at most covers y" to mean: "x covers y or $x = y$." Then modify (ξ') to: "if x and y at most cover a, then $x \cup y$ at most covers x and y"; this change makes no difference since the cases $x = y$, $x = a$, and $y = a$ are trivial. With this modified covering in mind, form connected chains

$$x \cap y = x_0 < \cdots < x_m = x, \qquad x \cap y = y_0 < \cdots < y_n = y.$$

Then assuming by induction that $x_{i-1} \cup y_j$ and $x_i \cup y_{j-1}$ at most cover $x_{i-1} \cup y_{j-1}$, we can conclude from the modified (ξ') that $x_i \cup y_j = (x_{i-1} \cup y_j) \cup (x_i \cup y_{j-1})$ at most covers $x_{i-1} \cup y_j$ and $x_i \cup y_{j-1}$. We infer that $d[x \cup y_j] - d[x \cup y_{j-1}] \leqq 1$, and hence $d[x \cup y] - d[x] \leqq n = d[y] - d[x \cap y]$, which proves the first assertion. The second follows by duality.

50. Modular functionals. Measure, probability, and dimension are familiar instances of *functionals* (real-valued functions) defined on lattices. In order to analyze the properties of such functionals, one wants

Definition 3.4:† *A functional $m[x]$ defined on a lattice is called "modular" if and only if*

M1: *$m[x] + m[y] = m[x \cup y] + m[x \cap y]$ identically.*

It is called "positive," if and only if

M2: *$x \geqq y$ implies $m[x] \geqq m[y]$.*

* Their sufficiency was first proved by the author [1], Thm. 10.2. Cf. ibid., §§8–9, for the other results of this section.

† The first two parts of the definition go back to Dedekind [2] implicitly; cf. also the author [1], Cor. 9.2. The general definition of "bounded variation" was first given by the author (Bull. Am. Math. Soc., 44 (1938), p. 186).

It is called "of bounded variation," if and only if the sums $\sum_{i=1}^{n} | m[x_i] - m[x_{i-1}] |$ $[x_0 < x_1 < \cdots < x_n]$ *are bounded.*

Probability and measure are always positive and modular; dimension is positive and often modular; any functional on a chain is modular. Definition 3.4 contains as special cases the usual definitions (Saks [1], pp. 18, 148) of bounded variation, both for ordinary real functions and for functions of sets. Also, any positive functional on a lattice with O and I is of bounded variation (since $\sum_{i=1}^{n} | m[x_i] - m[x_{i-1}] |$ is always bounded by $m[I] - m[O]$).

THEOREM 3.9: *The modular functionals on a modular lattice of finite dimensions are the linear combinations* $m[x] = \sum_p c_p d[p, x] + C$ *of the "prime dimension functions"* $d[p, x]$; $m[x]$ *is positive if and only if every* $c_p \geqq 0$.

Proof: Each $d[p, x]$ is modular by Theorem 3.3, rewritten $d[p, u \smile v] - d[p, u] = d[p, v] - d[p, u \frown v]$. And any linear combination of modular functionals is modular; hence every $\sum_p c_p d[p, x] + C$ is modular. Conversely, suppose $m[x]$ is modular, and let $m[x/y]$ denote $m[x] - m[y]$. Then M1 is the assertion that $m[x/y]$ is equal on transposes—and hence on projective quotients. Let c_p denote $m[x/y]$ on the prime quotient p and projective quotients; then clearly

$$m[x] = m[O] + m[x/O] = m[O] + \sum_p c_p d[p, x]$$

and we conclude that there are no other modular functionals. Finally, if $m[x]$ is positive, by definition the $c_p \geqq 0$; the converse is obvious since the $d[p, x]$ are positive.

51. Metric lattices. By a "quasi-metric lattice," we mean a lattice L with a positive modular functional $m[x]$. We shall call $m[x \smile y] - m[x \frown y] = m[x \smile y/x \frown y]$ the "quasi-distance"* between x and y, and shall denote it $\delta(x, y)$.

If in particular

M*2: $x > y$ *implies* $m[x] > m[y]$,

then we shall refer to L as a "metric lattice," to $m[x]$ as "sharply positive," and to quasi-distance as "distance."

Through these definitions, we can apply the metric ideas of Fréchet and his followers to lattice theory. Thus

* The author does not know who first used a "measure distance," but V. Glivenko [1] was the first to define quasi-distance abstractly. J. von Neumann [2], Chap. XVII was the first to discuss the uniform continuity of joins and meets in his "rank distance," although the author [6], Thm. 30 had done it for Boolean algebras.

Many of the results of §§51–4 have been obtained independently by M. F. Smiley and L. R. Wilcox, *Metric lattices*, Annals of Math., 40 (1939), 309–27. Cf. also V. Glivenko [2] and F. Maeda, *Lattice functions and lattice structure*, Jour. Sci. Hiroshima Univ., 9 (1939), 85–104.

LEMMA: *The transformations $x \to a \smile x$ and $x \to a \frown x$ are contractions; in fact*

M*: $\delta(a \smile x, a \smile y) + \delta(a \frown x, a \frown y) \leqq \delta(x, y)$.

Proof: The left-hand side of M* is by definition

$$m[a \smile x \smile y] - m[(a \smile x) \frown (a \smile y)] + m[(a \frown x) \smile (a \frown y)] - m[a \frown x \frown y]$$

which is, by the one-sided distributive law, at most

$$m[a \smile x \smile y] - m[a \smile (x \frown y)] + m[a \frown (x \smile y)] - m[a \frown x \frown y].$$

Transposing the middle terms, and using M1, this becomes $m[a] + m[x \smile y] - m[a] - m[x \frown y] = \delta(x, y)$.

THEOREM 3.10: *Quasi-distance is metric (cf. F) except that condition (2) is replaced by (2′) if $x \neq y$, $\delta(x, y) \geqq 0$. Joins and meets are uniformly continuous in the quasi-distance; hence if we identify x and y whenever $\delta(x, y) = 0$, we get a lattice-homomorphism of L onto a metric lattice.*

Proof: Fréchet's condition (1) follows from L1, (2′) from M2, (3) from L2. Again, clearly

$$\begin{aligned}\delta(x, y) + \delta(y, z) &= m[x \smile y/y] + m[y/x \frown y] + m[y \smile z/y] + m[y/y \frown z] \\ &= m[y \smile z/y] + m[x \smile y/y] + m[y/y \frown z] + m[y/x \frown y] \\ &\geqq m[x \smile y \smile z/x \smile y] + m[x \smile y/y] + m[y/y \frown z] + m[y \frown z/x \frown y \frown z]\end{aligned}$$

by M*. But this sum is evidently $m[x \smile y \smile z/x \frown y \frown z]$, and so is at least $\delta(x, z)$, proving (4).

But now M* shows that a displacement of the right-hand variable in $a \smile x$ and $a \frown x$ displaces the join resp. meet through at most as great a quasi-distance. By L2, the same is true of the left-hand variable, proving uniform continuity. Again, by (1), (2′), (3), and (4), the relation $\delta(x, y) = 0$ is reflexive, symmetric and transitive; by uniform continuity, it preserves joins and meets—hence it defines a lattice-homomorphism. Finally, in the image lattice, $x > y$ implies that $\delta(x, y) = m[x] - m[y] > 0$; hence the image lattice is a metric lattice.

We note that, generally speaking, $x \neq y$ if and only if $x \smile y > x \frown y$; hence $x \neq y$ implies $\delta(x, y) > 0$ if and only if $u > v$ implies $m[u] > m[v]$. That is, the quasi-distance of Theorem 3.10 is metric, and the homomorphism an isomorphism, if and only if L is a metric lattice.

52. Modularity of metric lattices. The phrase "modular functional" gains in appropriateness through

THEOREM 3.11: *Any metric lattice is modular.*†

Proof: In any lattice, $x \leqq z$ implies $x \smile (y \frown x) \leqq (x \smile y) \frown z$ (one-sided modular law), hence it suffices to exclude the case $x \smile (y \frown z) < (x \smile y) \frown z$ by

† Cf. Dedekind [2], p. 268, and the author [1], Thm. 10.1.

proving $m[x \smile (y \frown z)] - m[(x \smile y) \frown z] = 0$. But by condition M1, this difference is

$$m[x] + m[y \frown z] - m[x \frown y \frown z] - m[x \smile y] - m[z] + m[x \smile y \smile z].$$

And since $x \leqq z$, this reduces after rearrangement to

$$(m[y \frown z] + m[y \smile z] - m[z]) - (m[x \smile y] + m[x \frown y] - m[x]),$$

which is by M1 simply $m[y] - m[y] = 0$, q.e.d.

COROLLARY 1: *A lattice of finite dimensions is modular if and only if its dimension function is modular.*

For we have already shown that the dimension function of any modular lattice is modular. Also, using Corollary 3 of Theorem 3.1 and the Lemma of §49, we get

COROLLARY 2: *A lattice of finite dimensions is modular if and only if it satisfies the covering conditions* (ξ')–(ξ'').

Now combining Theorems 3.9 and 3.10, we see that to any subset of prime quotients of a modular lattice of finite dimensions corresponds a congruence relation annulling those and only those prime quotients. Hence by Theorem 3.5,

THEOREM 3.12: *The congruence relations on any modular lattice of finite dimensions correspond one-one to the subsets of prime quotients (projective quotients being identified), the correspondence being with the quotients annulled. Hence they form a Boolean algebra.*

53. Continuous metric lattices. As was observed in the Foreword, any metric space is implicitly topologized by its distance function. This suggests asking: can one correlate the metric topology of a lattice with its order topology, or metric completeness in the sense of Cauchy-Fréchet* with "completeness" in the sense of §21.

The answer is that one can, provided $x_n \uparrow x$ implies $m[x_n] \uparrow m[x]$ and dually—that is, provided the functional $m[x]$ is *continuous* in the order topology. We shall call metric lattices in which these dual conditions hold "continuous metric lattices."

LEMMA 1: *In order that a metric lattice be continuous it is necessary and sufficient that (o)-convergence imply metric convergence.*

Proof: If (o)-convergence implies metric convergence, then $x_n \uparrow x$ implies

* A metric space is called "complete" if and only if every "fundamental sequence" $\{x_n\}$ (sequence such that $\delta(x_m, x_n) \to 0$ as $m, n \to \infty$) "converges" to a point x of the space. Fréchet showed that Cantor's algorithm for obtaining the real number system from the rationals could be applied to any metric space; MacNeille's Theorem shows that likewise Dedekind's process can be applied to any partially ordered system.

$\delta(x, x_n) = m[x] - m[x_n] \to 0$ and dually. Conversely, in a continuous metric lattice, if $u_n \uparrow x$, $v_n \downarrow x$, and $u_n \leqq x_n \leqq v_n$, then

$$\delta(x, x_n) = m[x \smile x_n/x \frown x_n] \leqq m[v_n/x] + m[x/u_n],$$

and this converges to zero by hypothesis.

Corollary: *It is also necessary and sufficient that star-convergence imply metric convergence.*

Proof: It is sufficient by Lemma 1, since (o)-convergence implies star-convergence. Conversely, if $\{x_n\}$ star-converges to x, then every subsequence of $\{x_n\}$ contains a subsubsequence (o)-converging to x, and thus converging to x metrically; hence (by a well-known lemma on metric convergence) $x_n \to x$ metrically, q.e.d.

Observe that the real number system is a *continuous* metric lattice and that a measure function is "completely additive" in the sense of Borel if and only if it is "continuous" in our sense. We shall recur to the definition of continuity when we discuss von Neumann's "continuous geometries" (Chapter IV) and Kolmogoroff's postulates for probability (Chapter VIII).

We could easily distinguish between upper and lower semi-continuity, but it is not obvious how useful this would be.

54. Metric topology vs. order topology. The metric topology of a continuous metric lattice is shown to be closely related to the order topology by

Theorem 3.13: *Let L be any continuous metric lattice with O and I. Then the following are equivalent: metric completeness, (order) completeness, and σ-completeness. If L is metrically complete, then star-convergence and metric convergence are equivalent.**

Proof: Suppose L is metrically complete, and let S be any subset of L. Consider the joins sup X of the finite subsets X of S; sup $\{m[X]\} \leqq m[I]$ will exist, and so we can find X_n such that $m[\sup X_n] \geqq \sup \{m[X]\} - 2^{-n}$. Then, letting U denote the set-union of X_m and X_n, we have

$$\delta(\sup X_m, \sup X_n) = m[\sup U/\sup X_m] + m[\sup U/\sup X_n] \leqq 2^{-m} + 2^{-n},$$

and so (by metric completeness) the sup X_n converge metrically to some s.

Now let $x \in S$ be given; by the Lemma of §51,

$$m[x \smile s/s] = \lim_{n\to\infty} m[x \smile \sup X_n/\sup X_n] \leqq 2^{-n}$$

for all n. Hence $x \smile s = s$, and s is an upper bound to S. While if u is any upper bound to S, then $u \frown \sup X_n = \sup X_n$ for all n, and so $u \frown s = s$ by

* These results were evolved jointly by J. von Neumann and the author (cf. Annals of Math., 38 (1937), p. 56). The equivalence of star-convergence and metric convergence was also shown by Kantorovitch [1]. It is a corollary of Theorem 3.13 that L is a topological lattice, since for monotonic sequences star-convergence and (o)-convergence are equivalent.

continuity (§51). We conclude that $s = \sup S$, whence (by duality) metric completeness implies (order) completeness.

Now drop the assumption of metric completeness, and suppose $\delta(x_m, x_n) \to 0$ as $m, n \to \infty$. Then a subsequence $\{y_k\} = \{x_{n(k)}\}$ of $\{x_n\}$ exists, with $\delta(y_k, y_{k+h}) \leqq 2^{-k}$. If L is σ-complete, then lim sup $\{y_k\}$ exists, and its distance from y_m is (by continuity) the limit as $n \to \infty$ of

$$\delta(y_m, \sup \{y_{n+k}\}) \leqq \delta(y_m, y_n) + \sum_{k=1}^{\infty} m[y_n \smile \cdots \smile y_{n+k}/y_n \smile \cdots \smile y_{n+k-1}].$$

Each term under the summation sign is at most $\delta(y_{n+k}, y_{n+k-1})$ by the Lemma of §51, and so the sum is at most 2^{1-n} —which tends to zero as $n \to \infty$. We conclude $\delta(y_m, \limsup \{y_k\})$ is at most 2^{-m}, whence $\{y_m\}$, and so $\{x_n\}$, converges to lim sup $\{y_k\}$ metrically. Thus σ-completeness implies metric completeness. But by definition (order) completeness implies σ-completeness; hence all three are equivalent.

In virtue of the Corollary of §53, it only remains to prove that metric convergence implies star-convergence. But this can be deduced from the last paragraph, which shows that any metric limit is the lim sup of a subsequence—dually, it is the lim inf—hence a metric limit is the (o)-limit of a subsequence, and thus of a subsubsequence of every subsequence. We conclude: a metric limit is a star-limit, q.e.d.

But now, in any metric abstract algebra whose operations are uniformly continuous functions of their arguments, the Cantor-Meray* process of metric completion permits one and only one extension of operations leaving them continuous. Hence we obtain as a direct corollary of Theorem 3.10,

THEOREM 3.14: *A metric lattice has a unique metrically complete hull, in which it is (metrically) dense.*

The reader should be cautioned that, although a continuous metric lattice is complete if and only if it is metrically complete, it does not follow that for continuous metric lattices metric completion yields the *same* complete lattice as completion by cuts. We shall verify this later (Theorems 6.15–6.18).

55. Jordan's decomposition. Jordan's decomposition of functions of bounded variation into monotone summands was generalized by Riesz (Saks [1], p. 8) to additive set-functions. We shall make a further generalization, which will apply to modular functionals on any lattice L.

First recall that if $m[x]$ and $m_1[x]$ are two modular functionals on L, then so is their difference $m[x] - m_1[x]$. We *define* the relation $m \geqq m_1$ to mean that $m[x] - m_1[x]$ is positive; the relation evidently satisfies P1 and P3.

What Jordan really did was to look for least upper and greatest lower bounds of modular functionals; we can give almost verbatim repetitions of many parts

* Cf. Hausdorff [1], p. 106. The extension of operations is that used in defining sums and products of irrationals.

of his construction. Thus with any chain γ: $y = x_0 < x_1 < \cdots < x_n = x$, we shall associate the "positive variation" of $m[x]$ on γ, namely,

$$m^+[x/y, \gamma] = \sum_{i=1}^{n} \sup \{m[x_i/x_{i-1}], 0\}.$$

We shall define $m^+[x/y]$ as sup $m^+[x/y, \gamma]$, and shall call it the "positive variation of $m[x]$ between x and y." This always exists if $m[x]$ is of bounded variation between x and y.

Lemma 1: $m^+[x/y] = m^+[x/t] + m^+[t/y]$.

Proof: Clearly $m^+[x/t] + m^+[t/y]$ is the supremum of the positive variations of $m[x]$ of those chains between x and y which pass through t—and so is at most sup $m^+[x/y, \gamma]$ for all chains. It is therefore sufficient to show that if γ: $y = x_0 < x_1 < \cdots < x_n = x$ is any chain, then there is a chain γ_t through t with $m^+[x/y, \gamma] \leqq m[x/y, \gamma_t]$; this will prove the reverse inequality. But consider

$$\gamma_t\colon 0 = x_0 \frown t \leqq x_1 \frown t \leqq \cdots \leqq x_n \frown t = x_0 \smile t \leqq x_1 \smile t$$
$$\leqq \cdots \leqq x_n \smile t = x.$$

By M1, $m[x_i \smile t] = m[x_i] + m[t] - m[x_i \frown t]$, and likewise for $m[x_{i-1} \smile t]$. Substituting and cancelling, we get

$$m[x_i \smile t/x_{i-1} \smile t] = m[x_i] - m[x_i \frown t] - m[x_{i-1}] + m[x_{i-1} \frown t],$$

whence $m[x_i \smile t/x_{i-1} \smile t] + m[x_i \frown t/x_{i-1} \frown t] = m[x_i/x_{i-1}]$. It is now quite easy to see that $m^+[x/y, \gamma] \leqq m^+[x/y, \gamma_t]$, which is what we wanted to show.

Lemma 2: $m^+[x \smile y/x] = m^+[x/x \frown y]$.

Proof: Let γ: $x \frown y = x_0 < x_1 < \cdots < x_n = x$ be given; form γ^*: $y = x_0 \smile y \leqq x_1 \smile y \leqq \cdots \leqq x_n \smile y = x \smile y$. Then as in Lemma 1, $m[x_i \smile y/x_{i-1} \smile y] = m[x_i] - m[x_i \frown y] - m[x_{i-1}] + m[x_{i-1} \frown y]$, which is $m[x_i/x_{i-1}]$ since $x_i \frown y = x \frown y = x_{i-1} \frown y$. We infer $m^+[x/x \frown y, \gamma] = m^+[x \smile y/y, \gamma^*]$, whence $m^+[x/x \frown y] \leqq m^+[x \smile y/y]$. The reverse inequality is obtained by a dual argument, completing the proof.

Now recall that the relation $m \geqq m_1$ does not satisfy P2; more precisely, Theorem 1.2 identifies functionals differing by a constant. But if we admit only functionals which vanish on some distinguished element a such as O (as is the case with measure, probability, and with additive functionals on linear lattices—cf. Chapter VII), we get just one element from each residue class, and this trouble disappears. With this convention, we can prove

Theorem 3.15: *Given $m[x]$, sup $\{m, 0\}$ exists and is*

$$m^+[x] = m^+[x \smile a/a] - m^+[x \smile a/x].$$

Proof: Clearly $m^+[a] = 0$. Again, $m^+[x]$ is modular. For if $x \geqq y$, then by definition

$$m^+[x] - m^+[y] = m^+[x \smile a/a] - m^+[x \smile a/x] - m^+[y \smile a/a] + m^+[y \smile a/y].$$

By Lemma 2, $m^+[y \smile a/y] = m^+[a/a \frown y]$ and $m^+[y \smile a/a] = m^+[y/a \frown y]$. Substituting, and applying Lemma 1 to eliminate a, we get

$$m^+[x] - m^+[y] = m^+[x \smile a/a \frown y] - m^+[x \smile a/x] + m^+[y/a \frown y].$$

And this, by Lemma 1 applied to the chain $x \smile a \geqq x \geqq y \geqq a \frown y$, leaves a remainder of $m^+[x/y]$. In summary, $m^+[x] - m^+[y]$ equals $m^+[x/y]$. Modularity is a corollary of this and Lemma 2.

Now that the modularity of $m^+[x]$ has been established (this is trivial if L is a chain—Jordan's case), the proof that $m^+[x] = \sup\{m, 0\}$ can be copied from Jordan. First, clearly $m^+[x/y] \geqq m[x/y]$ and 0 identically; hence m^+ is *an* upper bound to m and 0. Again, if $u \geqq m, 0$, then for all γ, $u[x/y] \geqq m^+[x/y, \gamma]$, and so $u[x/y] \geqq m^+[x, y]$ for all x, y. That is, $m^+[x]$ is a *least* upper bound to m and 0.

56. Connected subsystems of a modular lattice. Let L be any modular lattice of infinite dimensions, and divide the graph of L into its connected components. We shall show that the partition L thus defined is *homomorphic*. It will follow that the connected subsystems are individually convex sublattices.†

Indeed, if x covers x', then $x/(x' \smile y) \frown x$ and $x \smile y/x' \smile y$ are transposable. But the former is x/x or x/x'; hence by perspectivity either $x \smile y = x' \smile y$ or $x \smile y$ covers $x' \smile y$, and in either case $x \smile y$ and $x' \smile y$ are in the same connected subsystem. By induction, if x and x' are connected, then so are $x \smile y$ and $x' \smile y$. While duality and L2 take care of the other four cases.

Ore calls the image-lattice the "once reduced" lattice obtained from L; it is evident that the process can be iterated as long as the image-lattice remains disconnected.

This and related notions play an important role in the theory of polynomial ideals, and hence in algebraic geometry. We shall not consider this application here.

57. Projectivity and operator-isomorphism. Let G be any group, with or without operators. Denote the (possibly void) set of its operators by Ω, and denote by Ω^*, the set Ω augmented by the inner automorphisms of G. It is known (van der Waerden [1], pp. 134–5) that the congruence relations on G correspond one-one to its Ω^*-subgroups; it follows by Theorem 3.2 and §43 that they form a modular lattice.‡

We shall now correlate quotients, perspectivity, and projectivity with important group-theoretic concepts. To do this, we shall need some further concepts. If M and N are two Ω^*-subgroups of G, and $M \geqq N$, then we can define M/N as the quotient-group (i.e., quotient-algebra in the sense of the

† These observations are due to Ore [1], pp. 421–4.

‡ Incidentally, if A is any abstract algebra, and A_θ any quotient-algebra of A, then the congruence relations on A_θ form a lattice which is isomorphic with the sublattice of those congruence relations on A, which "include" θ.

Foreword) of the cosets of N in M. We shall also say that M/N and M_1/N_1 are "operator-isomorphic," if and only if there exists a one-one correspondence between their elements which preserves group multiplication, and multiplication by every operator of Ω^* as well.†

THEOREM 3.16: *Let L be the lattice of Ω^*-subgroups of any group, with or without operators. Projective quotients in L correspond to operator-isomorphic quotient-groups.*

Proof: Since operator-isomorphism is symmetric and transitive, it suffices to show that $M \smile N/M$ and $N/M \frown N$ are always operator-isomorphic. Now assume the calculus of complexes, whereby XY denotes the set of products $xy\ [x \in X,\ y \in Y]$. Consider the correspondence $Mx \to Mx \frown N\ [x \in N]$. Since $Mx \frown N = Mx \frown Nx = (M \frown N)x$, this carries elements of $M \smile N/M$ into elements of $N/M \frown N$. Since $M(M \frown N)x = Mx$, it is one-one (has a single-valued inverse). Finally, since

$$MxMy = Mxy \to (M \frown N)xy = (M \frown N)x(M \frown N)y$$

while $(Mx)\omega = (M\omega)(x\omega) = M(x\omega) \to [(M \frown N)x]\omega = (M \frown N)x\omega$ similarly, it is an operator-isomorphism.

Theorem 3.16 applies to quotient-groups, quotient-rings, quotient-algebras of linear algebras, quotient-spaces of linear spaces, and to reductions of group representations and ring representations.

58. The Jordan-Hölder Theorem. Combining Theorem 3.16 with Theorem 3.6, we get the generalized theorem of Jordan-Hölder,‡

THEOREM 3.17: *In Theorem 3.16, the quotient-algebras between successive terms of any connected chain of congruence modules are unique, to within operator-isomorphism and their arrangement in a sequence.*

It has been proved by A. Kurosch and the author, working independently, that the same conclusion holds for well-ordered *ascending* series of congruence modules—but not for descending ones.§ We omit the proof.

† With groups, operator-isomorphism is usually called "central isomorphism"; it is an isomorphism preserved under all inner automorphisms. With rings and linear algebras, it is an isomorphism preserved under left- and right-multiplication by all elements; in representation theory, it is an isomorphism between quotient-spaces preserved under all transformations.

‡ Due originally to C. Jordan, *Commentaire sur Galois*, Math. Ann., 1 (1869), 141–60 (with numerical interpretation), and O. Hölder, *Zurückführung einer beliebigen Gleichung* ···, ibid., 34 (1889), 26–56. The generalized point of view appears to be due to E. Noether and W. Krull. For the corresponding theorem on composition series, cf. E. George, *Über den Satz von Jordan-Hölder-Schreier*, Crelle's Jour., 180 (1938), 110–20; A. I. Uzkow, *On the Jordan-Hölder Theorem*, Math. Sbornik, 46 (1938), p. 42.

§ A. Kurosch, Math. Ann., 111 (1935), 13–8, and the author, Bull. Am. Math. Soc., 40 (1934), 847–50. The non-duality is due to the fact that if $N_\alpha \uparrow N$, then $M \frown N_\alpha \uparrow M \frown N$, whereas the dual of this need not hold. Cf. §102.

59. Free modular lattice with three generators. We shall now prove that the free modular lattice generated by three symbols x_1, x_2, x_3 has twenty-eight elements, and exhibit its Hasse diagram in Fig. 5. This result is due to Dedekind.*

It is clear by Corollary 2 of Theorem 3.11 that the lattice of Fig. 5 is modular. Further, it is generated by the x_i; in fact $u_1 = x_2 \smile x_3$ cyclically, $v_1 = x_2 \frown x_3$

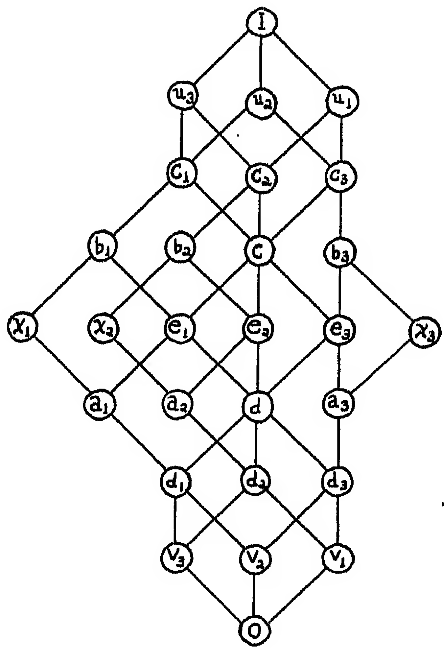

FIG. 5

and cyclically, $I = x_1 \smile x_2 \smile x_3$, $O = x_1 \frown x_2 \frown x_3$, $a_i = x_i \frown u_i$, $b_i = x_i \smile v_i$, $c_1 = u_2 \frown u_3$ and cyclically, $d_1 = v_2 \smile v_3$ and cyclically, $c = u_1 \frown u_2 \frown u_3$, $d = v_1 \smile v_2 \smile v_3$, and $e_i = u_i \frown (x_i \smile v_i) = (u_i \frown x_i) \smile v_i$.

Finally, the binary join- and meet-functions in the lattice diagrammed are consequences of L1–L5. The calculations involved in proving this are so numerous, even when reduced by cyclic symmetry and duality, that we shall only give two samples:

$$a_1 \smile a_2 = (x_1 \frown u_1) \smile (x_2 \frown u_2) = ((x_1 \frown u_1) \smile x_2) \frown u_2 \qquad \text{by L5}$$

$$= (u_1 \frown (x_1 \smile x_2)) \frown u_2 = u_1 \frown u_3 \frown u_2 = c,$$

* R. Dedekind [2]; this paper includes a table of joins and meets, reproduced in different notation by the author [1]. Ore [1], p. 414 suggests the diagram; the author [6], p. 443 maps it on lattice-homomorphic images of two and five elements.

$$b_2 \cup e_3 = (x_2 \cup v_2) \cup (a_3 \cup v_3) = x_2 \cup a_3 \qquad \text{by absorption}$$

$$(\text{since } v_3 \leqq x_2 \quad \text{and} \quad v_2 = x_3 \cap x_1 \leqq x_3 \cap u_3 = a_3)$$

$$= x_2 \cup (x_3 \cap (x_1 \cup x_2)) = (x_2 \cup x_3) \cap (x_1 \cup x_2) = c_2.$$

These samples are typical, in that they use L5 as a *mixed associative law*.

We conclude that the lattice of Fig. 5 *is* the free modular lattice with three generators, which thus has twenty-eight elements.

On the other hand, the free modular lattice generated by four elements is infinite. Indeed, consider the modular lattice of subspaces of three-space (lines, planes, etc.) passing through the origin. If two perpendicular axes are chosen through the origin, and two nearly parallel planes through each axis, then the four planes generate an infinite sublattice.*

60. Free modular lattice generated by two chains. Let L be any lattice, and let $O = x_0 < x_1 < \cdots < x_m = I$ and $O = y_0 < y_1 < \cdots < y_n = I$ be any two chains in L between O and I. Clearly the set of $u_j^i = x_i \cap y_j$ includes all x_i and y_j (for $x_i \cap y_n = x_i$ and $x_m \cap y_j = y_j$); dually, the set of $v_j^i = x_i \cup y_j$ includes them. Hence so does the set of *joins* of the u_j^i, and that of the *meets* of the v_j^i.

LEMMA 1: *Any join of the* u_j^i *can be written in the form* $(x_{i(1)} \cap y_{j(1)}) \cup \cdots \cup (x_{i(r)} \cap y_{j(r)})$, *where* $i(1) > \cdots > i(r)$ *and* $j(1) < \cdots < j(r)$.

Proof: If two u_j^i have the same superscript, then since the y_j are a chain, one u_j^i must be contained in, and hence by L4 can be absorbed by, the other. Thus we can make all the $i(k)$, and similarly all the $j(k)$, distinct. Moreover if $i > i'$ and $j \geqq j'$, then $(x_i \cap y_j)$ will absorb $(x_{i'} \cap y_{j'})$, since it includes it. Hence after we have absorbed as many elements as possible, and utilized L2 to arrange the $i(k)$ in descending order, we shall have $j(1) < \cdots < j(r)$ also.

LEMMA 2: *If* $a_i \geqq a_{i+1}$ *and* $b_i \leqq b_{i+1}$ *for all* i *in a* modular *lattice, then*

$$(a_1 \cap b_1) \cup \cdots \cup (a_r \cap b_r) = a_1 \cap (b_1 \cup a_2) \cap \cdots \cap (b_{r-1} \cup a_r) \cap b_r,$$

$$(b_1 \cup a_1) \cap \cdots \cap (b_r \cup a_r) = b_1 \cup (a_1 \cap b_2) \cup \cdots \cup (a_{r-1} \cap b_r) \cup a_r.$$

Proof: By duality and induction on r, we need only prove the first identity on the assumption that the second holds when there are fewer than r summands. But by L5 applied twice, $(a_1 \cap b_1) \cup \cdots \cup (a_r \cap b_r)$ can be rewritten in the following form:

$$a_1 \cap [b_1 \cup (a_2 \cap b_2) \cup \cdots \cup (a_{r-1} \cap b_{r-1}) \cup a_r] \cap b_r.$$

And by the second identity for the case $(r - 1)$, the two expressions

$$(b_1 \cup a_2) \cap (b_2 \cup a_3) \cap \cdots \cap (b_{r-1} \cup a_r),$$

$$b_1 \cup (a_2 \cap b_2) \cup (a_3 \cap b_3) \cup \cdots \cup (a_{r-1} \cap b_{r-1}) \cup a_r$$

* This example was discovered by the author [1], p. 464.

are equal. And if we substitute the former for the latter in the square brackets above, we get the right-hand side of the first identity, q.e.d.

LEMMA 3: *The joins of the u_j^i are a sublattice.*

Proof: Evidently any join of joins of u_j^i is a join of u_j^i; but any meet of joins of u_j^i is by Lemmas 1–2 a meet of meets of v_j^i, hence a meet of v_j^i, and hence by Lemmas 1–2 a join of u_j^i.

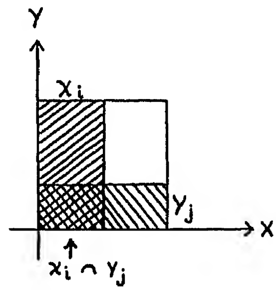

FIG. 6

Now observe that if X_i denotes the set of points (x, y) (see Fig. 6) of the unit square $0 \leqq x, y \leqq 1$ satisfying $x \leqq i/m$, if Y_j denotes the set of points satisfying $y \leqq j/n$, and if joins and meets are interpreted as set-unions and set-products, then all of the expressions admitted in Lemma 1 describe different sets (of saw-tooth shape). Hence these sets form by Lemma 3 a *ring* of sets (i.e., a sublattice of the modular lattice of all subsets of the square), representing isomorphically the *free* modular lattice generated by the given chains. In the notation of §16, the lattice is $B^{C_m C_n}$. We infer that the free modular lattice generated by any two chains is distributive in the sense of Chapter V, and

THEOREM 3.18: *The different elements of the free modular lattice generated by two chains are the different expressions admitted in Lemma* 1, *and form a lattice isomorphic with a ring of sets.*

THEOREM 3.19 (of Schreier-Zassenhaus):* *Let $\{X_i\}$ and $\{Y_j\}$ be any two chains of congruence modules in a group (with or without operators). Then the chains can be refined by interpolation, so as to make their factors operator-isomorphic in pairs.*

Proof: By Theorem 3.18 and Lemma 2, the chains generate a finite modular lattice. Refining $\{X_i\}$ and $\{Y_j\}$ into (finite) chains connected in *this*, the result follows from Theorem 3.17, applied to this sublattice.

Alternative proof, not assuming Theorems 3.17–3.18: Form the chains having as quotients $X_i \smile (Y_{j+1} \frown X_{i+1})/X_i \smile (Y_j \frown X_{i+1})$ and $Y_j \smile (X_{i+1} \frown Y_{j+1})/Y_j$

* O. Schreier, *Über den J-H'schen Satz*, Abh. Hamb., 6 (1928), 300–2. H. Zassenhaus, *Zum Satz von J-H-S*, ibid., 10 (1934), 106–8.

$\smile (X_i \frown Y_{j+1})$; these will be refinements of $\{X_{i+1}/X_i\}$ and $\{Y_{j+1}/Y_j\}$ respectively. Moreover (lemma of Zassenhaus) the two quotients are *projective*; indeed, each is *perspective* to $(X_{i+1} \smile Y_j) \frown (X_i \smile Y_{j+1})/(X_i \smile Y_j)$. To prove the first perspectivity (the second follows by symmetry in X and Y), we need only observe that, by L5,

$$[X_i \smile (Y_{j+1} \frown X_{i+1})] \smile (X_i \smile Y_j) = X_i \smile (Y_{j+1} \frown X_{i+1}) \smile Y_j$$
$$= X_i \smile [Y_{j+1} \frown (X_{i+1} \smile Y_j)] = (X_i \smile Y_{j+1}) \frown (X_{i+1} \smile Y_j)$$

and

$$(X_i \smile Y_j) \frown [(X_i \smile Y_{j+1}) \frown X_{i+1}] = (X_i \smile Y_j) \frown X_{i+1} .$$

Theorem 3.19 contains Theorem 3.17 as a corollary.

61. Subdirect and direct decompositions. Let S be any subalgebra of a direct union $A_1 \times \cdots \times A_r$ of abstract algebras A_i. Then S is a subalgebra of the direct union $S_1 \times \cdots \times S_r$ of the subalgebras S_i of elements of A_i appearing as ith components of elements of S. We shall say S is a "subdirect union" of the S_i.

The correspondence from each element $a = [a_1, \cdots, a_r]$ of S to its ith component a_i is clearly a *homomorphism* $\theta_i : S \to S_i$. Moreover if θ_i is regarded as a *congruence relation* on S, then two elements are congruent modulo every θ_i if and only if they are identical, component by component. Hence $\theta_1 \frown \cdots \frown \theta_r = 0$.

Conversely, given congruence relations $\theta_1, \cdots, \theta_r$ on an abstract algebra S, to define the residue class mod θ_i which contains any $a \in S$ as the ith *component* a_i of a, maps S homomorphically onto a subalgebra of the direct union $S_1 \times \cdots \times S_r$ of the algebras S_i of residue classes of S mod θ_i. This is an isomorphism if and only if $\theta_1 \frown \cdots \frown \theta_r = O$. Hence

THEOREM 3.20: *The representations of any abstract algebra A as a subdirect union correspond one-one to the sets of congruence relations on A satisfying $\theta_1 \frown \cdots \frown \theta_r = O$.*

There is no equally general theorem yielding the representations of A as a direct union. But in the special case of groups with or without operators, there is such a theorem—and curiously enough, it also applies to the complemented modular lattices which will be studied in Chapter IV.

THEOREM 3.21: *A representation of a group* G (with or without operators) as a subdirect union is* direct *if and only if the θ_i are* independent *under I.*

Proof: By induction, we can reduce to the case of two congruence modules. But if $M \frown N = 1$, $M \smile N = G$, then the xy $[x \in M, y \in N]$ form a group since $(xy)(x'y') = (xx')(x'^{-1}yx'y^{-1})(yy')$; moreover $x'^{-1}yx'y^{-1}$ is in M quâ $x'^{-1}(yx'y^{-1})$

* The assumption that G is a group is essential; in general, there is no lattice-theoretic criterion which tells which subdirect decompositions are direct.

and in N quâ $(x'^{-1}yx')y^{-1}$, whence $x'^{-1}(yx'y^{-1}) = 1$ and $(xy)(x'y') = (xx')(yy')$. Finally, $xy = x'y'$ implies $x'^{-1}x = y'y^{-1} \in M \frown N$, whence $x = x'$, $y = y'$. In summary, G consists of the different $[x, y] = xy$, where $[x, y][x', y'] = [xx', yy']$ and $[x, y] = (xy)\omega = [x\omega, y\omega]$.

62. A six-way quotient isomorphism. Now go back to the figure of §48, and observe that

$$e_1 \frown e_2 = e_2 \frown e_3 = e_3 \frown e_1 = d, \qquad e_1 \smile e_2 = e_2 \smile e_3 = e_3 \smile e_1 = c. \tag{62.1}$$

Hence e_1/d and e_3/d are both perspective to c/e_2, and so by Theorem 3.16 operator-isomorphic. Similarly e_2/d is operator-isomorphic to e_1/d, and by Theorem 3.21, c/d is the direct union of any two of these.

But if $G \times H$ is the direct union of two operator-isomorphic groups, then inner automorphisms induced by elements of G, which automorphically leave every element of H invariant, must leave every element of G invariant—i.e., G must be Abelian. Similarly, if G and H are rings or hypercomplex algebras, then since $gH = 0$ for any $g \in G$, we see that $gG = 0$, and so $GG = 0$. We conclude

THEOREM 3.22: *Let X_1, X_2, X_3 be any three congruence modules in a group G (with or without operators). Further, let $E_1 = (X_2 \smile X_3) \frown [X_1 \smile (X_2 \frown X_3)]$, and cyclically, let $D = (X_1 \frown X_2) \smile (X_2 \frown X_3) \smile (X_3 \frown X_1)$ and let C be dual to D. Then the six quotients E_i/D and C/E_i are operator-isomorphic, and C/D is the direct union of any two of them.*

In particular, the E_i/D are Abelian groups; and if G is a ring or hypercomplex algebra, then multiplication is trivial in the sense that all products* are 0.

63. Theorem of Kurosch-Ore. Theorem 3.20 makes one interested in the representations of an element (namely, O) of a modular lattice as a meet of larger elements. Ideal theory also involves such representations.†

Now we shall call an element a of a modular lattice L "meet-reducible" (for short, "reducible") if it is the meet $x \frown y$ of elements $x > a$, $y > a$ greater than itself; otherwise we shall call it "irreducible." A simple inductive argument shows that if L satisfies the ascending chain condition, then every $a \in L$ has a representation as a meet of irreducible elements.

* Historical note: The fact that if e_2 is a subgroup, if e_1 and e_3 are normal subgroups, and if (62.1) holds, then e_1/d and e_3/d are equal goes back to Jordan; for their isomorphism, cf. O. Bolza, *On the construction of intransitive groups*, Am. Jour., 11 (1889), 195–214. R. Remak, Crelle's Jour., 162 (1930), 1–16, pointed out that e_2 was normal if and only if the isomorphism was "central," and hence the factors had to be Abelian. Dedekind [2], p. 248, formula (31), gives the projectivities involved, but treats quotients as numbers only (i.e., he uses Jordan's but not Hölder's part of the Jordan-Hölder Theorem). The author [1], p. 462 and [2], p. 119 got Remak's result with Dedekind's method. Cf. also Ore, Duke Jour., 3 (1937), p. 174.

† Partly because any "irreducible" ideal (cf. infra) is "primary" in the sense of containing a power of a "prime" ideal. Cf. van der Waerden [1], Chap. XII.

Again, it is natural to call a component x_k in a "reduction" $a = x_1 \frown \cdots \frown x_r$ of a "redundant," if $a = x_1 \frown \cdots \frown x_{k-1} \frown x_{k+1} \frown \cdots \frown x_r$ —and obvious that to any reduction of a there corresponds one, none of whose components is redundant. Such reductions will be called "irredundant."

Lemma: *Let $a = x_1 \frown \cdots \frown x_r = x_1^* \frown \cdots \frown x_s^*$ be any two irredundant reductions of a into irreducible components. Then one can substitute for any x_i a suitable x_j^*, and get a new reduction of a.*

Proof: Set $y_i = x_1 \frown \cdots \frown x_{i-1} \frown x_{i+1} \frown \cdots \frown x_r$. Then by irredundancy,† $y_i > a$, yet $x_i \frown y_i = a$. Now form $z_j = y_i \frown x_j^*$; clearly $y_i \geqq z_j \geqq a$ —and, since $z_j \leqq x_j^*$, $a \leqq z_1 \frown \cdots \frown z_s \leqq x_1^* \frown \cdots \frown x_s^* \leqq a$. But by Theorem 3.3, the sublattice between $a = x_i \frown y_i$ and y_i is isomorphic to the sublattice between x_i and $x_i \smile y_i$ —and since x_i is irreducible in the latter, so is a in the former. Hence some z_j is a, and $x_1 \frown \cdots \frown x_{i-1} \frown x_j^* \frown x_{i+1} \frown \cdots \frown x_r = a$.

Theorem 3.23:‡ *The number of components in irredundant reductions of any element is independent of the reduction: in the preceding Lemma, $r = s$.*

Proof: Choose r minimal, and replace the x_i by x_j^* one at a time. We get in the end $a = x_{j(1)}^* \frown \cdots \frown x_{j(r)}^*$, whence by irredundancy of the x_j^* there are at least s $j(i)$, and $s \leqq r$. Hence by minimality, $s = r$.

Corollary: *Let G be any group with or without operators whose congruence modules satisfy the ascending chain condition. The number of factors in any irredundant representation of G as a subdirect union of groups not further decomposable is independent of the representation.*

64. A Theorem of Ore. An even more important result, due in its lattice-theoretic form to Ore,§ is

Theorem 3.24: *Let L be any modular lattice of finite dimensions. If $x_1, \cdots, x_r$ and $y_1, \cdots, y_s$ are any sets of elements independent under I whose meet is O, and if no x_i or y_j is the meet of larger independent elements, then the I/x_i and I/y_j are projective in pairs (whence $r = s$).*

Note that the x_i and y_j are not assumed to be meet-irreducible—and that

† Incidentally, since $y_1 \smile \cdots \smile y_k < x_{k+1}$, the y_i are independent over a. But we shall not need this fact.

‡ Historical note: Theorem 3.23 was proved for ideals by Emmy Noether, *Idealtheorie in Ringbereichen,* Math. Ann., 83 (1921), 24–66, §3. A different proof for finite groups was given by R. Remak, Crelle's Jour., 163 (1930), 1–44. The first lattice-theoretic proof was given by A. Kurosch [1]; O. Ore [2], p. 270 gave a proof independently.

§ O. Ore [2], p. 272. The theorem was first stated for finite groups by J. H. M. Wedderburn, *On the direct product in the theory of finite groups,* Annals of Math., 10 (1909), p. 173—although Kronecker (Berl. Sitz., 1870, 881–9) proved it for Abelian groups. A rigorous proof was given by R. Remak, Crelle's Jour., 139 (1911), p. 293. Recent papers by Krull, Schmidt, Fitting, and Korinek had extended the theorem to groups with operators.

representations as the meet of *independent* elements are to direct unions of groups as representations as meets alone are to subdirect unions.

We shall omit the proof of Theorem 3.24—it depends on a delicate inductive argument—and shall only point out its

COROLLARY: *The representation of any group (with or without operators) as a direct union of direct-indecomposable factors is* unique *to within pairwise operator-isomorphism of the factors, provided the congruence modules form a lattice of finite dimensions.**

* We note here the existence of a considerable literature on "residuated lattices." This analyzes lattices of ideals, using ideal multiplication as an auxiliary operation. Cf. notably M. Ward and R. P. Dilworth, *Residuated lattices*, Trans. Am. Math. Soc., 45 (1939), 335–54, and *Evaluations over residuated structures*, Annals of Math., 40 (1939), 328–38; also R. P. Dilworth's Doctoral Thesis, *The structure and arithmetical theory of non-commutative residuated lattices.*

The reader's attention is also called to the existence of many papers by Ore, applying lattice theory to groups. Cf. for example *The theory of groups*, Duke Jour., 5 (1939), 431–60, and the references given there. But these belong rather to group than to lattice theory. The author regrets being unable to give a more adequate account of these researches, owing to his having finished Chapters I–IV in May, 1938.

CHAPTER IV

COMPLEMENTED MODULAR LATTICES

65. Definition. The present chapter will be concerned with "complemented" modular lattices, in the sense of*

DEFINITION 4.1: *A (modular) lattice is "complemented" if and only if it has a O and I and*

L7: *Every x has a "complement" x', such that $x \frown x' = O$ and $x \smile x' = I$.*

THEOREM 4.1:† *Each of the following conditions is necessary and sufficient that a modular lattice of finite dimensions be complemented, and so is the dual of each condition.*

L7$_1$: *Given $a \leqq x \leqq b$, then x has a "relative complement" y satisfying $x \frown y = a$, $x \smile y = b$.*
L7$_2$: *Every element is the join of atoms.*
L7$_3$: *I is the join of atoms.*

Proof: We shall prove the implications L7 $\rightarrow$ L7$_1$ $\rightarrow$ L7$_2$ $\rightarrow$ L7$_3$ $\rightarrow$ L7; because of finite-dimensionality, we can assume the existence of O and I. Given $a \leqq x \leqq b$,

$$(a \smile x') \frown b \frown x = (a \smile x') \frown x = a \smile (x' \frown x) = a,$$

$$x \smile a \smile (x' \frown b) = x \smile (x' \frown b) = (x \smile x') \frown b = b,$$

and so $(a \smile x') \frown b = a \smile (x' \frown b)$ is the relative complement we want. Again, if $O < x < a$ and L7$_1$ holds, then a is the join of x and its relative complement y in a/O; hence by induction a is the join of atoms, proving L7$_2$. The implication L7$_2$ $\rightarrow$ L7$_3$ is trivial when I exists, and so it remains to prove that L7$_3$ implies L7.

But setting $x = x_0$, we can construct a chain between x_0 and I as follows. If $x_k < I$, then by L7$_3$ there exists a p_{k+1} not contained in x_k; set $x_{k+1} = x_k \smile p_{k+1}$. Some x_r will be I; define $x' = p_1 \smile \cdots \smile p_r$. Evidently $x \smile x' = x \smile p_1 \smile \cdots \smile p_r = I$. Moreover using the dimension function (cf. Theorem 3.18, Corollary), $d[x \frown x'] = d[x'] - d[I/x]$. But $d[x'] \leqq \sum_{k=1}^{r} d[p_k] = r$ and $d[I/x] \geqq r$; hence $d[x \frown x']$ is at most zero, completing the proof.

* In combination with distributivity, condition L7 goes back to Boole [2], pp. 49–50. In combination with modularity and other hypotheses, it was studied by K. Menger [2]. The first study of complemented modular lattices per se was made by the author [4]. Much has been done by J. von Neumann; cf. also O. Ore [2], Chap. 3.

† The implication L7 $\rightarrow$ L7$_2$ was stated by the author [4], p. 745; the implication L7$_3$ $\rightarrow$ L7$_2$ in [1], Thm. 11.4. The implication L7 $\rightarrow$ L7$_1$ was stated by von Neumann [2], Thm. 1.3. The full theorem was given by Ore [2], Chap. 3, §1.

It is a corollary that a set of linear operators is "fully reducible" if every invariant subspace has an invariant complement.

Since the hypothesis of finite dimensions was not used until induction on a was mentioned, we have also the

COROLLARY: *In any complemented modular lattice,* L7$_1$ *holds and all join-irreducible elements are atoms.*

In order to see the irredundancy of two hypotheses of Theorem 4.1, note that the non-modular lattice of five elements satisfies L7 but not L7$_1$; hence we needed to assume modularity. Again, in "generalized Boolean algebras" (cf. infra), L7$_1$ holds but no I exists; hence we needed to assume finite-dimensionality.

Incidentally, the implication L7 $\rightarrow$ L7$_2$ is an immediate corollary of the following interesting

LEMMA: *The "differences" $d_k = x_k' \frown x_{k+1}$ between successive terms of any chain $O = x_0 < x_1 < \cdots < x_s$ are independent elements whose join is x_s.*

Proof: By induction on s, we can assume $(d_1 \smile \cdots \smile d_{s-1}) \smile d_s$ is $x_{s-1} \smile (x_{s-1}' \frown x_s)$. But by L5, this is $(x_{s-1} \smile x_{s-1}') \frown x_s = x_s$. The d_k are independent by induction and since $(d_1 \smile \cdots \smile d_{s-1}) \frown d_s = x_{s-1} \frown x_{s-1}' \frown x_s = O$.

66. Examples. Since set-complements satisfy L7, the algebra of all subsets of any class I forms a complemented modular lattice; for the same reason, so does any field of subsets of I.

Using L7$_3$, we see that the linear subspaces of any linear space, the normal subgroups of any direct union of simple groups, and the ideals (resp. invariant subalgebras) of the direct sum of any finite set of simple rings (resp. linear algebras) form complemented modular lattices. We shall see in §78 that any abstract projective geometry is a complemented modular lattice.

The notion of a complemented modular lattice appears also in representation theory. Let Ω be any group or algebra of linear operators, operating on a linear space I. Then the condition that the modular lattice of Ω-subspaces of I be complemented is by definition that each "half-reduction" of I by an Ω-subspace X, be part of a "full reduction" of I by X and a complementary Ω-subspace X'—and by Theorem 4.1 that I be the sum of "irreducible" components. These conditions and their equivalence are well-known in algebra. It is a fundamental theorem that they hold if Ω is a finite group or a semi-simple hypercomplex algebra.*

Since L7 is self-dual, the dual of any complemented modular lattice is itself complemented. Moreover since $x \frown x' = y \frown y' = O$ and $x \smile x' = y \smile y' = I$

* At least, unless the characteristic of the field of scalars for I divides the order of Ω. Cf. Wedderburn, *Lectures on Matrices*, American Mathematical Society Colloquium Publications, vol. 17, New York, 1934, pp. 165-9. With groups and *real* scalars, there is a simple proof: let $q(x)$ denote the sum of the transforms of any positive definite quadratic form by Ω; then $q(x)$ is invariant under Ω, and so the set X' of perpendiculars to X is an Ω-subspace, and (by positive definiteness) satisfies L7.

clearly imply $[x, y] \frown [x', y'] = [O, O] = O$ and dually, any product of complemented lattices is itself complemented. Again, L7 shows that any lattice-homomorphic image of a complemented modular lattice is itself one. And finally, by $L7_1$, if a and b are any fixed elements of a complemented modular lattice, then the sublattice $L(a, b)$ of elements between a and b is also complemented.

It is a corollary that every sub-representation of a "fully reducible" representation is fully reducible.

67. Quotients. One can eliminate the notion of "quotients" x/y from the theory of complemented modular lattices, by calling the "difference" $y' \frown x = x - y$ between y and x a "representative" of x/y, and identifying quotients with their (transpose) representatives. Clearly $y' \frown x$ is a relative complement of y in x.

In the new theory, the notion of "transposable quotients" is replaced by that of "perspective elements," in the sense of

Definition 4.2: *Two elements a and b are "perspective" if and only if they have a common complement c, called an "axis of perspectivity" for a and b.*

Remark: As was first pointed out by von Neumann ([2], p. 19), this generalizes a fundamental definition of projective geometry—as does the notion of projectivity. The key property of projective geometries is that any two equidimensional elements are perspective; this does not hold in complemented modular lattices which are not projective geometries.

Theorem 4.2: *Two elements a and b are connected by a sequence of perspectivities if and only if a/O and b/O are projective.*

Proof: If a and b are perspective by c, then a/O, I/c and b/O are transposable in that order; hence $a/O \sim b/O$, and by induction this remains true if a and b are connected by a sequence of perspectivities. Conversely, any relative complement c of $u \frown v$ in u is a relative complement* of v in $u \smile v$; moreover any two relative complements $(u \smile v) - v$ are perspective by $v \smile (u \smile v)'$. Hence *any* two "representatives" of transposable quotients are perspective and any two representatives of projective quotients are connected by a sequence of perspectivities.

68. Congruence relations. Let L be any lattice with O. Then the residue classes under any congruence relation θ on L are convex sublattices and in particular, the residue class J_θ of elements congruent to O mod θ is an "ideal" in the sense of†

Definition 4.3: *By an "ideal" of a lattice L is meant an M-closed sublattice: a subset which contains with any x and y, $x \smile y$, and with any x, all $x \frown t$ $[t \in L]$.*

* Proof: $c \frown v = (u \frown v)' \frown u \frown v = 0$, while $c \smile v$ is the join of c, $u \frown v$, and v—and hence is $u \smile v$.

† This definition is due to Stone [1]; cf. also Tarski, Fund. Math., 16 (1930), p. 181, and Moisil [1], p. 17. It is like the usual definition of an ideal in a commutative ring R, as a subset which contains, with any x and y, $x + y$, and with any x, all xr $[r \in R]$.

THEOREM 4.3: *In a complemented modular lattice, $x \equiv y\ (\theta)$ if and only if $x \smile y = (x \frown y) \smile t$ for some $t \equiv O\ (\theta)$. Thus θ is determined by the ideal J_θ of $t \equiv O\ (\theta)$.*

Proof: If $x \smile y = (x \frown y) \smile t \equiv (x \frown y) \smile O = x \frown y\ (\theta)$, then x and y, which lie between $x \frown y$ and $x \smile y$, are congruent. Conversely, if $x \equiv y\ (\theta)$, then $x \frown y \equiv x \smile y\ (\theta)$, and so the difference $t = (x \frown y)' \frown (x \smile y)$, which satisfies $t \smile (x \frown y) = x \smile y$ by the Lemma of §65, is congruent to $(x \frown y)' \smile (x \frown y) = O$.

69. Neutral elements. We now ask: which ideals are congruence modules? If the ascending chain condition holds, then clearly any ideal contains the join a of its members, and consists of the $t \leqq a$. In other words, it is the "principal ideal" $a \frown L$ of $a \frown u$ $[u \in L]$. This suggests the question: which $a \frown L$ are congruence modules J_θ?

To answer this, suppose $x \geqq a$, $y \geqq a$, and $x \equiv y\ (\theta)$. Then by Theorem 4.3, $x \smile y = (x \frown y) \smile t \leqq (x \frown y) \smile a = x \frown y$, and so $x = y$. That is, θ never identifies distinct elements which both include a. But it identifies each x with one $x \geqq a$, namely, $x \smile a$. We infer

THEOREM 4.4: *A principal ideal $a \frown L$ is a congruence module, if and only if the correspondence $x \rightarrow x \smile a$ is a lattice endomorphism.*

Elements a with this property in a modular lattice L are called "neutral."*

THEOREM 4.5: *The following conditions on an element a of a modular lattice are equivalent:* (1) *it is in the center of L,* (2) *it is complemented and neutral,* (3) *it has a unique complement.*

Proof: The implication (1) → (3) is contained in the Lemma of §31; we are thus proving the converse of this for modular lattices.

Indeed, suppose a has a unique complement a', and that $u \frown a = O$. Then by hypothesis u, a, and $(u \smile a)'$ are independent elements with join I; hence $u \smile (u \smile a)' = a'$ and $u \leqq a'$. Thus a' contains every u with $u \frown a = O$. In particular, since $[(a \frown x)' \frown x] \frown a = O$ irrespective of x, $(a \frown x)' \frown x$ is included in a' as well as x, and

$$x = (a \frown x) \smile [(a \frown x)' \frown x] \leqq (a \frown x) \smile (a' \frown x) \leqq x \smile x = x.$$

Hence $x = (a \frown x) \smile (a' \frown x)$, and every $x \in L$ can be written in the form

$$y \smile z \qquad [y \leqq a, z \leqq a'].$$

* The notion of "neutral" elements is due to Ore [1], pp. 419–21. Theorem 4.5 is due to J. von Neumann ([2], Part I, Thms. 5.3–5.4); the author collaborated in extending the proofs to the uncomplemented case. Another result on neutral elements is Theorem 6.1. For a complete theory of neutral elements, cf. the author's *Neutral elements in general lattices*, to appear in the Bulletin of the American Mathematical Society. N. B., one can profitably define an ideal in a complemented modular lattice to be "neutral," if and only if it contains with any element all perspective elements.

It follows from Theorem 3.4 that L is the product of the sublattices of $y \leqq a$ and $z \leqq a'$, which proves the implication (3) $\rightarrow$ (1).

It remains to prove the implications (1) $\rightarrow$ (2) and (2) $\rightarrow$ (3). But it is obvious that the correspondence $[x, y] \rightarrow [x, y] \smile [I, O]$, that is $[x, y] \rightarrow [I, y]$, is a lattice-endomorphism, proving (1) $\rightarrow$ (2). Finally, if a is neutral, and x and y are both complements of a, then $a \frown (x \frown y) = O$ and

$$a \smile (x \frown y) = (a \smile x) \frown (a \smile y) = I \frown I = I,$$

whence $x \frown y$ is a complement. But by Theorem 3.1, no complement can be *less* than x or y; hence $x = x \frown y = y$. Thus a has a unique complement, proving (2) $\rightarrow$ (3), q.e.d.

70. Fundamental decomposition theorem. We shall now prove a fundamental decomposition theorem on complemented modular lattices of finite dimensions.*

First recall that an abstract algebra is called "simple" if and only if it has no non-trivial congruence relations. This definition of universal algebra includes as special cases the usual definitions for groups (with and without operators), rings, and linear algebras.

THEOREM 4.6: *Let L be any complemented modular lattice of finite dimensions. The following five conditions are equivalent:* (a) *L is simple,* (b) *L is "indecomposable" (not a product),* (c) *all points of L are perspective,* (d) *all points of L are projective,* (e) *any two modular functionals on L are linearly dependent.*

COROLLARY: *Any finite-dimensional complemented modular lattice is the product of simple ones.*

We shall call simple complemented modular lattices of finite dimensions "projective geometries."

Proof: By Theorem 4.5, an element not O or I is neutral if and only if it is in the center; by Theorem 4.4 resp. definition, this means that L is simple if and only if it is indecomposable. Again, by Theorem 3.5, L is simple if and only if all its prime quotients are projective, hence by §67 if and only if (c) all its points are projective, and by Theorem 3.9, if and only if (e) any two modular functionals are linearly dependent. Since (c) trivially implies (d), it remains to prove (d) $\rightarrow$ (c).

For this, we need only show that projective points are perspective—and by induction we need only show that if p is perspective to q, and q to r, then p is perspective to r: that *perspectivity* is *transitive.* This will complete the proof of Theorem 4.6.

* This result is mainly due to the author (cf. [4], p. 747, and [6], Thm. 26, where (a) $\leftrightarrow$ (b) $\leftrightarrow$ (c) is shown; also Bull. Am. Math. Soc., 40 (1934), p. 209). It is also stated in Menger [3]. The relation of (b) to elements with a unique complement was shown by J. von Neumann.

71. Perspectivity is transitive. The proof that the relation of perspectivity between points is transitive is not hard.*

LEMMA: *Two points p and q are perspective if and only if $p \smile q$ contains a third point s.*

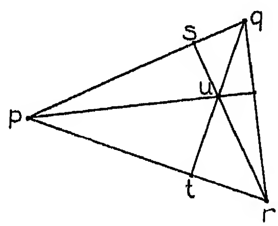

FIG. 7

Proof: Let a be the axis of perspectivity; then

$$d[a \frown (p \smile q)] = d[a] + d[p \smile q] - d[a \smile p \smile q] = d[I] - 1 + 2 - d[I],$$

and so $a \frown (p \smile q)$ is a third point on $p \smile q$. Conversely, if s is a third point on $p \smile q$, then, since p, s, and $(p \smile q)'$ are independent elements with join I, $c = (p \smile q)' \smile s$ is complementary to p; similarly, it is complementary to q, and so p and q are perspective.

Now let a be an axis of perspectivity for p and q, and b one for q and r. Let s and t be third points on $p \smile q$ resp. $p \smile r$. Then $u = (q \smile t) \frown (r \smile s)$ is by M1 a point; similarly, so is $(p \smile u) \frown (q \smile r)$ and in fact, a third point on $q \smile r$. Hence by the Lemma, perspectivity between points is a transitive relation. See Fig. 7.

This completes the proof of Theorem 4.6; it also makes it easy to prove the stronger result

THEOREM 4.7: *In a projective geometry, the following conditions are equivalent:* (a) $d[x] = d[y]$, (b) *x and y are perspective,* (c) *x and y are projective.*

Proof: The implications (b) → (c) → (a) are obvious, if we use M1. Now suppose $d[x] = d[y] = r$, and let $p_1, \cdots, p_r$ and $q_1, \cdots, q_2$ be bases for $x - x \frown y$ and $y - x \frown y$ respectively. Let $t_1, \cdots, t_r$ be third points on $p_1 \smile q_1, \cdots, p_r \smile q_r$. Then evidently

$$\begin{aligned} x \smile t_1 \smile \cdots \smile t_r &= (x \frown y) \smile p_1 \smile t_1 \smile \cdots \smile p_r \smile t_r \\ &= (x \frown y) \smile p_1 \smile q_1 \smile \cdots \smile p_r \smile q_r = x \smile y, \end{aligned}$$

* It was essentially given by the author [4], Lem. 2; perspective points were termed "conjoint." A proof of Theorem 4.7 was first given by J. von Neumann, in a much harder case. Cf. also I. Halperin, *On the transitivity of perspectivity in continuous geometries*, Trans. Am. Math. Soc., 44 (1938), 537-62.

and so by dimensionality, $x \frown y$, the p_i, the t_j, and $(x \smile y)'$ are independe elements whose join is I. Hence

$$(x \smile y)' \smile t_1 \smile \cdots \smile t_r, \qquad x = (x \frown y) \smile p_1 \smile \cdots \smile p_r$$

are complements; likewise, it and y are complements, whence x and y are pe spective, proving (a) $\to$ (b), q.e.d.

72. Projectivity of part with whole. Projectivity is a weak kind of equalit relation between quotients, and it is natural to try to make this equality mea equality in the sense of having equal magnitude.

Unfortunately, this assumption leads, in the case of some lattices, to th paradox of a whole's being equal to (presumably smaller) parts. Thus consid the lattice L of all subgroups of the direct union A of countable cyclic groups prime order. This is a complemented modular lattice, in which any two ind pendent infinite subgroups determine perspective elements. Hence an elemen of L may be projective to each of two independent parts—in spite of the fa that we would normally presume (by M1) that this would make its magnitu equal to the sum of the magnitudes of its parts.

Other exceptions are referred to by Ore [2], p. 274; their presence is howev impossible in metric lattices, by

THEOREM 4.8: *In a metric lattice, no quotient is projective with a "part" of itsel*

Proof: By this we mean that if $x \geqq u \geqq v \geqq y$ then x/y cannot be projectiv to u/v unless $u = x$, $v = y$. Indeed, $m[x/y] = m[u/v]$ —and in a metric lattic this is incompatible with $x > u$ or $v > y$.

COROLLARY: *In a modular lattice of finite dimensions, no quotient is projecti with a part of itself.*

We note also that by Theorem 3.1 an element cannot be perspective with part of itself; hence exceptions cannot occur in a complemented modular lattic if perspectivity is transitive. Again, exceptions cannot occur in distributiv lattices (cf. §97). Generally speaking, the existence of exceptions is closely tie up with the non-existence of dimension functions.

73. Semi-modular lattices. We shall call a lattice of finite dimensio "upper semi-modular" if and only if it satisfies condition (ξ') of Theorem 3. Corollary 3; dually, if it satisfies (ξ''), we shall call it "lower semi-modular. That semi-modularity does not imply modularity is exemplified by the lattic of all subgroups of the octic group* (Fig. 8)—also by the sublattice of shade elements.

* In fact, since (cf. Speiser, *Gruppentheorie*, second ed., Thm. 81) any maximal subgrou of a subgroup of order p^k is of order p^{k-1}, and two such subgroups intersect in one of orde p^{k-2}, the lattice of all subgroups of *any* group of prime-power order is lower semi-modula MacLane [1] cites the Boolean algebra of all subsets of a class with four members, in whic all elements of dimension exceeding two are identified. L. R. Wilcox has also treate semi-modularity in *Modularity in the theory of lattices*, Annals of Math., 40 (1939), 490–50

It is easy to show that any product, homomorphic image,* or convex sublattice of an upper semi-modular lattice is itself upper semi-modular; on the other hand, sublattices of semi-modular lattices need not be semi-modular (cf. Theorem 3.1). The dual of an upper semi-modular lattice is lower semi-modular and vice versa. We shall give other examples of semi-modular lattices in §77.

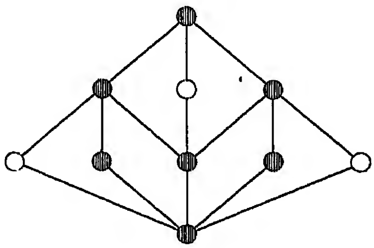

FIG. 8

We recall the results of §49 on upper semi-modularity—especially the Jordan-Dedekind chain condition, and the dimensional inequality

$$d[x] + d[y] \geqq d[x \frown y] + d[x \smile y].$$

Corollaries of these are: (1) the "exchange axiom" of Steinitz-MacLane: if p, q are points, and $a < a \smile p \leqq a \smile q$, then $a \smile p = a \smile q$, and (2) the second "exchange axiom" of MacLane: if p is a point, then either $p \leqq x$ or $p \smile x$ covers x.

We shall consider only *upper* semi-modular lattices in §§74–77; the theory of lower semi-modular lattices is dual.

74. Dependence and rank. We shall begin by obtaining an abstract theory of linear dependence in upper semi-modular lattices. For this we shall want various definitions.†

A sequence $x_1, \cdots, x_r$ of elements of an upper semi-modular lattice L will be called "independent" if and only if, for $k = 1, \cdots, r - 1$, $(x_1 \smile \cdots \smile x_k) \frown x_{k+1} = O$; a sequence which is not independent will be called "dependent." It is a corollary that any subsequence of an independent sequence is independent. Again, by the "rank" of a set S of points of L will be meant $d[\sup S]$; by a "basis"

* Proof: Let the homomorphism be $L \to L^*$, and suppose x^* and y^* cover a^* in L^*. Let a be the largest antecedent of a^*, and x and y arbitrary antecedents of x^* and y^*. Form chains from a to $a \smile x$ and $a \smile y$; the elements which cover a in each will be antecedents x_1 of x^* and y_1 of y^*. Hence $x_1 \smile y_1$ will be an antecedent of $x^* \smile y^*$, which must thus cover x^* and y^*. Thus even join-homomorphism is sufficient.

† H. Whitney [1] formulated an abstract theory of linear dependence, apparently unrelated to the inclusion relation. The author [5] reformulated Whitney's theory in terms of lattice theory. Cf. also T. Nakasawa, *Zur Axiomatik der linearen Abhängigkeit*, Reports Tokyo Bunrika Daigaku, 2 (1935), 235–55, and 3 (1936), 45–69, and 123–36, where many results are proved; *Über Abhängigkeitsräume*, by O. Haupt, G. Nöbeling, and Chr. Pauc, Jour. f. Math., 181 (1940), 193–217.

of an element $a \in L$ will be meant a set B of points whose join is a, and which forms an independent sequence no matter how ordered.*

LEMMA 1: *A sequence $p_1, \cdots, p_r$ of points is independent or dependent, according as its rank is r or less.*

Proof: Since $p_1 \smile \cdots \smile p_{k+1}$ at most covers $p_1 \smile \cdots \smile p_k$, we have $d[p_1 \smile \cdots \smile p_{k+1}] = d[p_1 \smile \cdots \smile p_k] + 1$ or $d[p_1 \smile \cdots \smile p_k]$ according as $(p_1 \smile \cdots \smile p_k) \frown p_{k+1} = O$ or not.

It is a corollary that the property of independence is preserved under all permutations†—that one can speak of independent and dependent *sets* of points (not just of sequences). Hence a basis of any $a \in L$ is any independent set of points whose join is a, and we have the further

COROLLARY: *The number of points in any basis of a is $d[a]$, and so an invariant of a, which is analogous to the Theorem of Kurosch-Ore.*

LEMMA 2: *Any set of points contains an independent subset of the same rank (i.e., with the same join m).*

Proof: We can construct term-by-term a sequence none of whose members is contained in the join of the preceding, until the join is m.

It is a corollary of this result and §66 that any normal subgroup of a direct union of simple groups is itself a direct union of simple groups (cf. van der Waerden [1], vol. 1, p. 143).

LEMMA 3 (Whitney's matroid postulate): *If $p_1, \cdots, p_s$ are independent, and if so are $q_1, \cdots, q_{s+1}$, then some set $p_1, \cdots, p_s, q_j$ is independent.*

Proof: By Lemma 1, $d[q_1 \smile \cdots \smile q_{s+1}] > d[p_1 \smile \cdots \smile p_s]$; hence some q_j is not in $p_1 \smile \cdots \smile p_r$.

THEOREM 4.9: *If $p_1, \cdots, p_n$ are independent points of L, then they generate a sublattice isomorphic with the field of all sets of p_i.*

Proof: Associate with each set S of p_i the join $a(S)$ of the $p_i \in S$. Clearly $a(S \smile T) = a(S) \smile a(T)$. Again, by Lemma 2, $d[a(S) \frown a(T)] = d[a(S)] + d[a(T)] - d[a(S \smile T)]$. But $d[a(X)]$ is the number of points in X, and so $d[a(S) \frown a(T)]$ is bounded (using the inequality) by the number of points in $S \frown T$, and so by $d[a(S \frown T)]$. But $a(S) \frown a(T) \geqq a(S \frown T)$ since it contains every $p_i \in S \frown T$—from which our dimensional inequality running the other way permits us to infer $a(S) \frown a(T) = a(S \frown T)$.

75. Reducibility conditions. We shall call a lattice L of finite dimensions "completely reducible" if and only if $L7_1$ holds.

Then any $a \in L$ not covering O is the join $p \smile y$ of any $p \leqq a$ and its relative

* It is easy to show that this is the case if and only if we have $x \frown \sup (B - x) = 0$ for all $x \in B$.

† Fr. Klein, *Birkhoffsche und harmonische Verbände*, Math. Zeits., 42 (1936), 58-81.

complement in a, so that L contains no irreducible elements except points. It is a corollary that ($L7_2$) every element of L is the join of points, and hence ($L7_3$) I is the join of points.

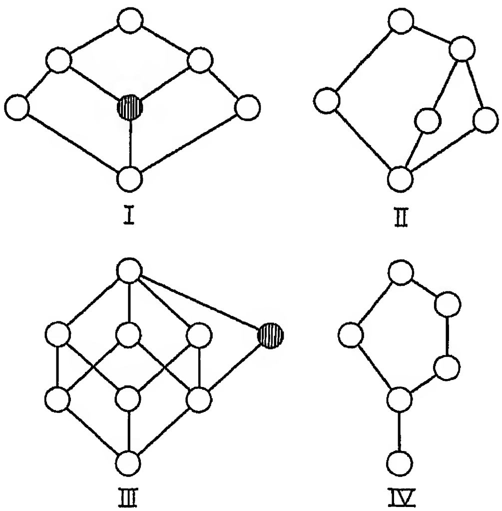

FIG. 9

Again, after deleting redundant points one-by-one, we can clearly express every element of L as the join of points which are symmetrically independent—i.e., every $a \in L$ has a *basis*.

On the other hand, without the modular axiom one cannot prove the reverse implications: in general, the conditions for "complete reducibility" just given are not equivalent, as is shown by the counterexamples of Fig. 9.

Thus in I, $L7_2$ holds, every $a \in L$ has a basis, Whitney's matroid postulate holds, and so any two bases for the same element have the same number of points. Although the dual of (ξ') holds, (ξ') itself does not. And what especially interests us, $L7_1$ does not hold, and Fr. Klein's condition does not hold, nor do MacLane's "exchange axioms." Again, II satisfies $L7_1$ and Whitney's matroid postulate, but not (ξ').*

76. Matroid lattices. By a "matroid lattice" or "exchange lattice," we shall mean† a completely reducible upper semi-modular lattice L.

We shall now show that we can replace $L7_1$ and (ξ') by apparently weaker consequences.

* This appears to contradict the assertion of the author in [5], p. 802, bottom. But Whitney's theory does not allow one to consider "asymmetrical" independence; hence Fr. Klein's condition is implicit, and by Lemma 2 of §66 we conclude that any lattice, arising from a "matroid," does not satisfy (ξ').

† The phrase "matroid lattice" was suggested by the author following Whitney; MacLane [1] speaks of "exchange lattices."

LEMMA 1: *Assuming* $L7_2$, *MacLane's first "exchange axiom" implies his second, and his second implies* (ξ').

Proof: If $x \smile q > z > x$, then some point p would be contained in z and not in x; hence $x < x \smile p < x \smile q$. Now suppose x and y cover a; we can write $x = a \smile p, y = a \smile q$, and hence $x \smile y = a \smile p \smile q$ covers both.

LEMMA 2: *From Fr. Klein's condition and* $L7_2$, *we can infer S. MacLane's first exchange axiom.*

Proof: Let $x_1, \cdots, x_n$ be a basis for a. Then unless $a \smile q \leqq a \smile p$, $(a \smile p) \frown q = O$, and so $x_1, \cdots, x_n, p, q$ are independent. Hence $x_1, \cdots, x_n, q, p$ are independent, and $p \leqq a \smile q$ is impossible.

LEMMA 3: *Condition* $L7_2$ *and* (ξ') *imply* $L7_1$.

Proof: Let $O < x < a$ be given; choose a basis p_i for x; then add new independent points one-by-one until a basis is had for a. Then by Theorem 4.9, the join of the new points will be a complement of x in a.

We can put the above results together, in

THEOREM 4.10: *A lattice is a "matroid lattice" if and only if it satisfies* (A) $L7_1$ *or* $L7_2$, *and* (B) *either of MacLane's exchange axioms, or Fr. Klein's condition, or* (ξ').

The results of §73 on products, homomorphic images, convex sublattices, general sublattices, and duals, of upper semi-modular lattices, remain true if one substitutes the word "matroid" for "upper semi-modular," throughout.

77. Applications. Many interesting examples of matroid lattices can be cited which are not modular.

For instance, the linearly closed manifolds of any affine space form a matroid lattice (Menger [3]); the proof of (ξ') and $L7_2$ is easy. In this example, the lattice dimension exceeds the ordinary (affine) dimension by one.

THEOREM 4.11: *The algebraically closed subfields of any field I form a matroid lattice.*

Proof: Let F be any algebraically closed subfield, and let H and K cover F. Choose x in H, y in K, neither in F. Then H is the set* of numbers algebraic over the ring $\{F, x\}$, K of those algebraic over $\{F, y\}$, and $H \smile K$ of those algebraic over $\{F, x, y\}$. Now if z is in $H \smile K$, then some polynomial $p(x, y, z) = 0$; if z is not in H then this polynomial must involve positive powers of y; hence y is algebraic over $\{F, x, z\}$. It follows that if $H < Z \leqq H \smile K$, then Z contains y, and so $Z = H \smile K$, which proves (ξ') and the Steinitz-MacLane

* We rely on the lemma: the set of numbers algebraic over any subring of I forms an algebraically closed subfield. For the algebraic background assumed, cf. van der Waerden [1], vol. 1, pp. 206-8. Theorem 4.11 and the applications of it are entirely due to S. MacLane [1].

"exchange axiom." Finally, the lattice satisfies $L7_2$: if $G > F$ choose a maximal set of numbers x_α of G which are algebraically independent over F; then the algebraic closures of the $\{F, x_\alpha\}$ will be points whose join is G.

We can define the "Transzendenzgrad" of G over F as the lattice-theoretic dimension of G over F. Then the above theorems on dependence, rank, etc., include as corollaries most of Steinitz' theory of algebraic dependence; for example, the number of x_α is $d[G/F]$. No satisfactory theory of *functional* dependence has yet been invented.

The lattice of all partitions† of any aggregate is also a matroid lattice; the lattice dimension expresses the number of parts into which the aggregate is divided. So is the lattice of all partitions of any graph into connected subgraphs; this is related to the four-color map theorem.

H. Whitney [1] has connected the theory of matroid lattices with vector dependence and with graphs. The connection with "schematic configurations" has also been pointed out by T. Nakasawa (op. cit.) and MacLane (*Some interpretations of abstract linear dependence*, Am. Jour., 58 (1936), 236–40).

Because of their connection with partitions, the study of matroid lattices should be important in combinatory analysis.‡

78. Projective geometries. We shall now show that the usual definition of a projective geometry is equivalent with the definition of §70.

Suppose L is a projective geometry in the sense of §70. Let us agree to call $(k+1)$-dimensional elements of L, simply "k-elements"; then 0-elements will be points—and we can call 1-elements "lines" and 2-elements "planes." Further, instead of writing $x \leqq y$, we shall say "x is on y." Then

PG1: *Two distinct points are on one and only one line.*

PG2: *If p, q, r are points not all on the same line, and if p^* and q^* are on $p \cup r$ and $q \cup r$, respectively, then $p \cup q$ and $p^* \cup q^*$ have a point s in common.*

PG3: *Every line contains at least three points.*

PG4: *Every $(k+1)$-element is of the form $p \cup a$ [p a point, a a k-element], and contains those and only those points q on lines through p and points of a.*

Proof: If $p \neq q$, then $d[p \cup q] = d[p] + d[q] - d[O] = 2$, proving PG1. Again, under the hypotheses of PG2, clearly $d[p \cup q \cup r] = 3$ and

$$d[(p \cup q) \cap (p^* \cup q^*)] = d[p \cup q] + d[p^* \cup q^*] - d[p \cup q \cup r] = 1,$$

and so $(p \cup q) \cap (p^* \cup q^*) = s$ is a point with the desired properties. PG3

† This lattice resembles the symmetric group of all permutations of the elements of the aggregate: its automorphisms are all induced by permutations of the elements, it has no non-trivial congruence relations (is "simple"), etc. Cf. Abstract 40-11-323 of the Bulletin of the American Mathematical Society; also the author [6], §§16-22.

‡ One would like to be able to answer any or all of the following questions: (1) is every finite modular lattice isomorphic with a lattice of a partitions? (2) is every (finite) lattice a sublattice of a matroid lattice?

Again, it is not certain what (ξ') should become in the infinite-dimensional case, although MacLane and Wilcox have done some work in this direction.

was proved in §70, while PG4 is obvious from $L7_2$ and the fact that $(q \smile p) \frown a$ is a point on a such that $[(q \smile p) \frown a] \smile p$ contains q by L5.

We conclude the

Lemma: *An n-dimensional projective geometry in the sense of §70 is an abstract $(n-1)$-dimensional projective geometry in the sense of Veblen and Young.**

79. Converse. This leaves us with the converse. Most of this is contained in the following result of Menger [2].

Lemma: *Any projective geometry is a complemented modular lattice. Meets are intersections, and the lattice dimension is the projective dimension plus one.*

Proof: It is obvious from PG4 that if we define $x \leqq y$ to mean "every point on x is on y," we get a partially ordered system. This is also obviously finite-dimensional, and the lattice dimension exceeds the projective dimension by one (using PG4 and induction). Also, intersections of elements of a projective geometry are again elements—hence the system is a lattice. Moreover $L7_3$ is obvious. It remains to prove L5, but this follows from the dimensional law $\dim L + \dim M = \dim L \frown M + \dim L \smile M$ (Veblen and Young, vol. 1, pp. 32–3, Thms. S_n2 and S_n3).

Theorem 4.12: *Our definition of a projective geometry is equivalent to the usual one.*†

Proof: After the last two lemmas, we need only show that all points in the second lemma are perspective. But this is precisely condition PG3 which thus becomes revealed as the condition that the complemented modular lattice defined by the other conditions be indecomposable.

Our definition is better than the usual one, in that it reveals the celebrated *duality principle* of projective geometry ab initio.‡ It reveals projective geometry immediately as "the geometry of section and projection" (its characterization since Th. Reye's *Geometrie der Lage*, 1866), and what used to be called "collineations" as simply *automorphisms* in the sense of universal algebra. It is also better in that it suggests the extension to continuous-dimensional (point-free) projective geometries.

80. Examples of projective geometries. Let F be any field or skew-field, and

* O. Veblen and J. W. Young, *Projective Geometry*, Boston, 1910. This Lemma is due to the author [4]. We note that PG2 is really just the condition that any two coplanar lines have a point in common.

† Remark: Menger has recently axiomatized affine and (real) non-Euclidean geometry (cf. [2] and *A new foundation of non-Euclidean, affine, real projective and Euclidean geometry*, Proc. Nat. Acad. Sci., 24 (1938), 486–90). He has shown (*Non-Euclidean geometry of joining and intersecting*, Bull. Am. Math. Soc., 44 (1938), 821–4) that order on a real line can be defined by lattice theory.

‡ The dual of $PG(F; n)$ is $PG(F^*; n)$, where F^* is anti-isomorphic to F. If in any theorem about n-dimensional projective geometries we replace k-elements by $(n-k)$-elements throughout, we get a true theorem.

denote by $V(F; n)$ the n-dimensional vector space over F. We shall show that the subspaces of $V(F; n)$ form a projective geometry, to be denoted $PG(F; n)$, which is n-dimensional in our sense.

Indeed, since $V(F; n)$ can be regarded as a group with operators, they form a modular lattice; this was proved earlier. This is evidently n-dimensional in our sense, since it has a basis of n elements. Further, I is evidently the join of "atoms," hence the lattice is complemented. It remains to show that there are three "points" on any "line." But let two points consist of the scalar multiples $c\xi$ and $c\eta$ of two vectors; then the projective "point" (in so-called homogeneous coordinates) of $c(\xi + \eta)$ is a third projective point on the line through them.

It is proved in Veblen and Young, by use of von Staudt's algebra of throws, that if $n > 3$, then the converse is true: any abstract projective geometry is a $PG(F; n - 1)$. The study of plane projective geometries not $PG(F; n - 1)$ takes us into one of the most mysterious and fascinating parts of combinatory analysis—a subject concerning which little is known.*

81. Continuous-dimensional projective geometries. We shall now construct continuous-dimensional projective geometries over any field F in the way indicated by von Neumann [1].

It is clear that if a and a' are complementary n-dimensional elements of $PG(F; 2n)$, then the sublattices X and Y of $x \leqq a$ and $y \leqq a'$ are isomorphic with $PG(F; n)$; hence they are isomorphic. Moreover since a and a' are independent, by Theorem 3.3 the joins $x \cup y\, [x \in X,\ y \in Y]$ form a *sublattice* isomorphic to $XX = X^2$, in $PG(F; 2n)$. But we can prove trivially the

LEMMA: *The couples $[x, x]$ of X^2 form a sublattice of X^2 which is isomorphic with X, under an isomorphism carrying O into O, I into I, and multiplying dimensions by two.*

We infer that $PG(F; n)$ can be embedded isomorphically in $PG(F; 2n)$ so that the "normalized dimension function" $d[x]/d[I]$ is preserved. Using iteration, we can construct a sequence of extensions of $PG(F; 1)$,

$$PG(F; 1) \to PG(F; 2) \to PG(F; 4) \to \cdots \to PG(F; 2^n) \to \cdots,$$

which preserve lattice operations and the "normalized dimension function" $d[x]/d[I]$.

Passing to the limit, we get an enveloping *metric lattice* containing elements of every "dimension" between zero and one, which can be expressed as a fraction $k/2^n$. By Theorem 3.14, this is embedded densely in a *complete* metric lattice containing elements of every dimension d, $0 \leqq d \leqq 1$. This last is the "continuous geometry" $CG(F)$ of von Neumann, having F as base field.† We omit proving that $CG(F)$ is complemented and irreducible.

* Rouse Ball's *Mathematical Recreations and Essays* contains some interesting material.

† An interesting construction analogous to von Neumann's has recently been suggested by Aronszajn and Glivenko (cf. Glivenko [2], p. 40).

82. Algebraic construction of dimension function. We know as a corollary of Theorem 3.13 that $CG(F)$ is a topological lattice—we shall omit the detailed proof.

Von Neumann has proved conversely that, in any complete, topological, complemented modular lattice without center, one can* introduce a dimension function, which becomes unique when "normalized" to make $d[O] = 0$, $d[I] = 1$.

To do this, he first defines $d[x] = d[y]$ as a relation which holds if and only if x and y are perspective. Then he proves that perspectivity is transitive, justifying his use of the equality notation. This leaves untouched the problem of correlating the purely abstract "dimension" with the real number system.

To accomplish this, he defines $d[x] \leqq d[y]$ to mean that x is perspective with a part of y. Then he proves that this relation simply orders his abstract "dimensional elements." Next he defines addition of "dimensional elements," by making $x \frown y = O$ imply $d[x \smile y] = d[x] + d[y]$. He gets in this way a truncated ordered Abelian group, and his last problem is to show that this is isomorphic either with the fractions having a fixed denominator n (the case of finite dimensions), or the continuum $0 \leqq x \leqq 1$ (the continuous-dimensional case).

83. Von Staudt's "algebra of throws." Not having so good a number system as we do, the Greeks performed arithmetic calculations on lengths with ruler and compass. It was an observation of von Staudt [1] that one can add, subtract, multiply and divide magnitudes with a ruler alone—that is, these operations are definable from the algebra of linear subspaces of the plane.

To see this, fix a "line at infinity" L and an "x-axis" X in the projective plane. The lines through $X \frown L$ will be the "parallels" $X_c : y = c$ to the x-axis, and perspectivity through any point on L defines a correspondence $(x, 0) \to (x + a, c)$ between X and X_c. Hence the most general translation of X can be accomplished by a sequence of two such perspectivities. And if we fix $(0, 0)$, we can *add* elements on X: letting p denote the intersection of L and the line through $(0, 0)$ and any point $(0, c)$ on X_c, and q that of L and the line through $(0, c)$ and $(a, 0)$, perspectivities through p and q in order will carry $(b, 0) \to (a + b, 0)$—and define the "sum" of a given $(a, 0)$ and $(b, 0)$.

Similarly, if we fix a y-axis Y, then perspectivity through any point on L will define a similarity transformation of X onto Y; reversing and using another axis of perspectivity, we get the most general expansion $(x, 0) \to (ax, 0)$ of X. Hence if we fix $(1, 0)$, we can *multiply* elements of X: given $(a, 0)$ and $(b, 0)$, we can construct the "product" $(ab, 0)$.

The amazing thing is that PG1–PG4 and Desargues' Theorem alone suffice to ensure that the "algebra of throws," so constructed, shall define a *field*.† This field is known to be commutative if and only if Pascal's Theorem holds.

* He assumes continuity hypotheses involving any increasing transfinite sequence.

† Cf. Veblen and Young [1], vol. 1, p. 140. Also D. Hilbert, *Grundlagen der Geometrie*, Chaps. V–VI. Also Veblen and Bussey, *Finite projective geometries*, Trans. Am. Math. Soc., 7 (1906), 241–59, and J. W. Young, *Projective Geometry*, Chicago, 1930, pp. 120–4.

Moreover if $d[I] \geqq 4$, Desargues' Theorem is known to be implied by PG1–PG4, but not otherwise.*

84. Coordinatization by "regular rings." Thus any projective geometry of finite dimensions $n > 3$ is isomorphic with the lattice of all subspaces of the n-dimensional linear space over a field F. Equivalently, as has been pointed out by von Neumann ([3], Part 2), it is the lattice of all left-ideals of the "simple" ring of all n^2-matrices over F.

Moreover this "ring-coordinatization" has an outstanding advantage over the usual "field-coordinatization" of von Staudt: it also coordinatizes reducible complemented modular lattices. It is easy to show† that the lattice of left-ideals of any sum R of simple rings $R_1, \cdots, R_n$ is the product of the lattices of left-ideals of the R_i individually. Hence in general, the finite-dimensional complemented modular lattices are the lattices of all left-ideals of "semi-simple" rings.

In this case, von Neumann amplifies Wedderburn's theory by some new results: two semi-simple‡ rings are isomorphic if and only if the lattices of their (principal) left-ideals are isomorphic, and anti-isomorphic if and only if the latter are dually isomorphic. Also, two-sided (principal) ideals, which correspond to idempotents in the "center" of the ring, likewise constitute the "center" of the lattice of left-ideals.

Also, it is made plain that Wedderburn's proof that any "semi-simple" ring is a direct sum of "simple" rings, while the latter are "full matrix rings" over fields F_i, corresponds to our theorem that any complemented modular lattice is a product of "projective geometries," while the latter are "full subspace lattices" over (von Staudt) fields F_i.

Ring-coordinatization also has the advantage that it can be extended in a natural way to the infinite-dimensional case. The construction (which requires the assumption that I/O can be divided into four mutually perspective parts) will not even be sketched; the reader will find it in the paper by von Neumann, cited above.

85. Orthocomplemented modular lattices. We have seen that some modular

* Cf. Veblen and Wedderburn, Trans. Am. Math. Soc., 8 (1907), 379-88. This suggests the problem of finding lattice-theoretic equivalents to Desargues' and Pascal's Theorems. But we can show that *no* lattice identity can be equivalent to Pascal's Theorem. For any identity true in a lattice is a fortiori true in a sublattice. And the three-dimensional projective geometry over the (non-commutative) quaternion field is contained as a sublattice in the twelve-dimensional projective geometry over the real field. On the other hand, any finite field is commutative (Wedderburn), and so this does not apply to the finite case.

† If J is any left-ideal, then $J = RJ \leqq (R_1 + \cdots + R_n)J \leqq R_1J + \cdots + R_nJ$; hence J is the sum of its components in the R_i. We now apply §30.

‡ Von Neumann [3] has introduced the word "regular" ring—and shown that regularity is equivalent to the assumption that every $a \in R$ have a "qualified inverse" x, such that $axa = a$. For further literature cf. F. Maeda, *Ring decomposition without chain condition,* Jour. Sci. Hiroshima Univ., 8 (1939), 145-67, and *Homomorphic basis for continuous geometry,* ibid., 9 (1939), 73-84.

lattices are "orthocomplemented," in the sense that a complementation is defined satisfying L7 and

L8: $(x \frown y)' = x' \smile y'$, $(x \smile y)' = x' \frown y'$, *and* $(x')' = x$.

Thus this holds for the lattice of all subspaces of Euclidean n-space,* if x^* denotes the orthogonal complement of x. It also holds for the subsets of any aggregate I, provided x' denotes the set-complement of x.

It is natural to ask which modular lattices of finite dimensions are orthocomplemented. It is easy to show that a reducible modular lattice is orthocomplemented, if and only if its irreducible factors are. This concentrates our interest on the n-dimensional projective geometries $PG(F; n)$ over (not necessarily commutative) fields F.

In this case, it is known† to be necessary and sufficient that F admit an anti-automorphism $x \to w(x)$ of period two, with a "definite diagonal Hermitian form" $\sum_{i=1}^{n} w(x_i)a_iy_i$, such that $w(a_i) = a_i$, and that $\sum_{i=1}^{n} w(x_i)a_ix_i = 0$ implies every x_i is zero.

86. Modular functionals. When we specialize to the complemented case, several definitions involving functionals become equivalent to simpler ones:

THEOREM 4.13: *Let* $m[x]$ *be any functional on a complemented modular lattice, and suppose* $m[O] = 0$. *Then* (a) $m[x]$ *is modular if and only if* $m[x \smile y] = m[x] + m[y]$ *when* $x \frown y = O$. *If modular, it is* (b) *of bounded variation if and only if bounded,* (c) *non-negative if and only if* $m[x] \geqq 0$ *identically, and* (d) *positive if and only if* $x > O$ *implies* $m[x] > 0$.

Proof: The necessity of the conditions is clear in all four cases. As to the sufficiency of (a), set $x - x \frown y = t$, $y - x \frown y = u$. Then t, u, and $x \frown y$ are independent by the lemma of §65, while $x \smile y = (x \frown y) \smile t \smile u$. Hence $m[x \frown y] = m[x \frown y] + m[t] + m[u]$, $m[x] = m[x \frown y] + m[t]$, and $m[y] = m[x \frown y] + m[u]$, from which M1 follows. As to (b), note that if we replace quotients by differences, we replace the positive variation by $m\left[\sum x_k - x_{k-1}\right]$, summed for the quotients on which the change is positive. Hence the positive variation is bounded by sup $|\, m[x]\,|$; so is the negative variation, whence the

* Under the same definitions, the (non-modular) lattice of all closed subspaces of Hilbert space is orthocomplemented. Cf. M. H. Stone, *Linear Transformations in Hilbert Space and their Applications to Analysis*, American Mathematical Society Colloquium Publications, vol. 15, New York, 1932, pp. 20-1.

† Cf. G. Birkhoff and J. von Neumann, *The logic of quantum mechanics*, Annals of Math., 37 (1936), p. 835 and Appendix. Zassenhaus, according to Artin, has shown that any dual automorphism of $PG(F; n)$ is in fact polarity with respect to a bilinear form, provided $n > 2$. The case in which L8 holds, but the complement of every point contains that point (the case of so-called "null systems"—cf. W. C. Graustein's *Geometry*) has been treated by R. Brauer, *A characterization of null systems in projective space*, Bull. Am. Math. Soc., 42 (1936), 247–54.

J. von Neumann ([2], Part III) has further related the existence of orthocomplements to that of "Hermitian" idempotents in the coordinatization by regular rings.

total variation is bounded by $2 \cdot \sup | m[x] |$. As to (c) and (d), simply note that if $x > y$, then $x - y > O$ and $m[x] = m[y] + m[x - y]$.

87. A decomposition theorem. Again, let $m[x]$ be any modular functional with $m[O] = 0$ on a complemented modular lattice L.

LEMMA 1: *The set J_0 of a such that $x \leqq a$ implies $m[x] = 0$ is a neutral ideal.*

Proof: Evidently it is M-closed; again, if $a, b \in J_0$, and $x \leqq a \smile b$, then we get

$$m[x] = m[x \frown a] + m[x/x \frown a] = m[x \frown a] + m[x \smile a/a] = 0 + 0$$

since $x \smile a/a$ is the transpose of a part of b. Thus J_0 is an ideal. Again, if a and a^* are perspective, then the sublattices of $x \leqq a$ and of $y \leqq a^*$ are metrically isomorphic; hence J_0 is a neutral ideal.

A similar argument shows that the set J^+ of a such that $O < x \leqq a$ implies $m[x] > 0$, and the dual set J^-, are also neutral ideals. Moreover J_0, J^+, and J^- are clearly without common element—hence (being neutral) they are independent.

Now suppose that L is topological and complete, and that $m[x]$ is "completely continuous" in the sense that $x_\alpha \uparrow x$ implies $m[x_\alpha] \to m[x]$ and dually,† where limits are understood in the generalized sense of §38. Then the join a_0 of the $a \in J_0$ will be a limit of finite joins of $a \in J_0$, and hence of elements of J_0. It follows, since L is topological, that any $x \leqq a_0$ will be the limit of $x \frown a$ $[a \in J_0]$, and so will satisfy $m[x] = \lim m[x \frown a] = 0$. We conclude: $a_0 \in J_0$—that J_0 is a principal ideal, generated by an element a_0 which (J_0 being neutral) must be in the center of L.

A similar argument shows that under the same hypotheses J^+ and J^- are also principal ideals, generated by elements a^+ and a^- also in the center of L.

Now consider the complement b of $a_0 \smile a^+ \smile a^-$. Since b is not in J_0, $m[x] \neq 0$ for some $x \leqq b$; suppose $m[x] > 0$. Deleting from x one at a time parts t on which $m[t] \leqq 0$, as long as possible, we get a residuum r such that (using modularity and continuity) $m[r] \geqq m[x] > 0$, and $t \leqq x$ implies $m[t] > 0$. But this would make $r > O$ in J^+—contrary to its definition as contained in the complement of a^+.

We conclude $b = O$ and $a_0 \smile a^+ \smile a^- = I$, whence

THEOREM 4.14: *If L is topological and complete, and $m[x]$ is completely continuous, then L is the product of three components—on which $m[x]$ is zero, positive, and negative, respectively.*

In other words, the decomposition of the variation of $m[x]$ (Jordan's decomposition) is associated with a decomposition of L itself.

It is a corollary that if L cannot be decomposed, as in the case of projective geometries, then any two modular functionals must be proportional: otherwise some linear combination of them would have mixed sign, and we could use Theorem 4.14 to obtain a decomposition.

† This cannot be unless $m[x]$ is of bounded variation.

CHAPTER V

DISTRIBUTIVE LATTICES

88. Definition. Many important lattices satisfy the following three identities,

L6: $(x \frown y) \smile (y \frown z) \smile (z \frown x) = (x \smile y) \frown (y \smile z) \frown (z \smile x)$.
L6′: $x \frown (y \smile z) = (x \frown y) \smile (x \frown z)$.
L6″: $x \smile (y \frown z) = (x \smile y) \frown (x \smile z)$.

Definition 5.1: *A lattice will be called "distributive" if and only if it satisfies* L6, L6′, L6″ *identically.**

Theorem 5.1: *Each of identities* L6, L6′, L6″ *implies* L5 *and both of the others.*†

Proof: Assuming L6 and $x \geqq z$, one gets by direct substitution $(x \frown y) \smile z = (y \smile z) \frown x$, which is L5 rearranged. Similarly, assuming L6′ and $x \geqq z$, one gets $x \frown (y \smile z) = (x \frown y) \smile z$. Finally, assuming $x \leqq z$ in L6″, one gets $x \smile (y \frown z) = (x \smile y) \frown z$. Hence each of the conditions L6, L6′, L6″ implies L5.

Now assuming L6″ we get by expansion,

$$[(x \frown y) \smile (y \frown z)] \smile (z \frown x)$$
$$= [(x \frown y) \smile (y \frown z) \smile z] \frown [(x \frown y) \smile (y \frown z) \smile x]$$
$$= [(x \frown y) \smile z] \frown [(y \frown z) \smile x] \qquad \text{by L3–L4}$$
$$= (x \smile z) \frown (y \smile z) \frown (y \smile x) \frown (z \smile x) \qquad \text{by L6″ again}$$
$$= (x \smile y) \frown (y \smile z) \frown (z \smile x) \qquad \text{by L2–L3}$$

which is L6. Conversely, abbreviating L6 to the form $u = v$, we get from the equality $x \smile u = x \smile v$, $x \smile (y \frown z)$ on the left-hand side, and on the right-hand side, using L5,

$$x \smile ([y \smile z] \frown [(x \smile y) \frown (x \smile z)]) = (x \smile y \smile z) \frown (x \smile y) \frown (x \smile z).$$

* Synonyms are "Dualgruppe von Idealtypus" (Dedekind), "distributiver Verband" (Fr. Klein), "arithmetic structure" (Ore).

† Historical note: It is curious that C. S. Peirce [1] should have thought that every lattice was distributive. He even said L6′, L6″ are "easily proved, but the proof is too tedious to give"! His error was demonstrated by Schröder [1], p. 282, who showed that L6′, L6″ were not implied by L1–L4, but (p. 286) implied each other and L6. A. Korselt (Math. Ann., 44 (1894), 156–7) gave another demonstration. Peirce at first [2] gave way before these authorities, but later (cf. E. V. Huntington [1], pp. 300–1) boldly defended his original view.

Dedekind [1], p. 116 showed that each of L6′ and L6″ implied L5. Menger [3], p. 480 showed that L6 implied L5 and L6′, L6″. We note also an important unpublished condition of J. Bowden (1936): $x \smile (y \frown z) \geqq (x \smile y) \frown z$, equivalent to distributivity.

But by L3–L4, this is $(x \smile y) \frown (x \smile z)$, the right-hand side of L6″. Thus L6″ and L6 are equivalent; dually L6′ and L6 are equivalent, completing the proof.

Corollary: *Any distributive lattice is modular.*

Remark: The implications of the preceding paragraph can also be read off from the figure of §59, using §62.

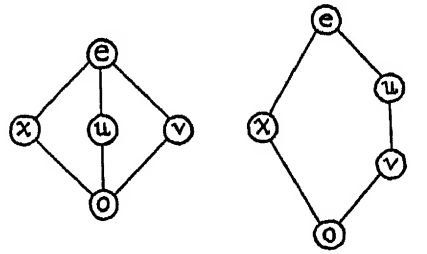

Fig. 10

From the same figure, it is evident that in a non-distributive modular lattice, the e_i form with c and d the sublattice graphed in Fig. 10. Combining with Theorem 3.1, we conclude

Theorem 5.2: *A lattice which is not distributive contains one of the examples of Fig.* 10 *as a sublattice.*†

Corollary 1: *A lattice is distributive if and only if relative complements in it are uniquely determined.*

This means that, given $a \leqq x \leqq b$, at most one y exists satisfying $x \frown y = a$ and $x \smile y = b$.

Proof: In a distributive lattice, $x \frown u = x \frown v$ and $x \smile u = x \smile v$ imply

$$u = u \frown (x \smile v) = (u \frown x) \smile (u \frown v)$$

$$= (v \frown x) \smile (u \frown v) = v \frown (u \smile x) = v.$$

Conversely, in a non-distributive lattice, the element x in either example of Fig. 10 has two relative complements.

Corollary 2: *Any non-distributive modular lattice which satisfies either chain condition contains the first example of Fig.* 10 *as a sublattice in which x, u, v cover o.*

Proof: If the descending chain condition holds, then we can find a $x^* \leqq x$ which

† G. Birkhoff [3], Thm. 4. Corollary 2 is Theorem 5, ibid. Corollary 1 is due to Bergmann [1], p. 273; cf. also Ore [1], p. 414, condition (δ_2). The condition is related to ideas of R. Grassmann [1]. Cf. Chap. VI.

covers O. Then $(x^* \smile u) \frown v$ and $(x^* \smile v) \frown u$ will provide the other examples; the role of e will be played by

$$[(x^* \smile u) \frown v] \smile x^* = (x^* \smile u) \frown (v \smile x^*).$$

The covering conditions are guaranteed by Theorem 3.3. The case of the ascending chain condition may be treated dually.

89. Examples. Any chain is a distributive lattice: in any chain, each side of L6 is the middle one of the elements x, y, z. Again, the subsets of any aggregate I form a distributive lattice—and more generally, any ring of sets is a distributive lattice. Thus the open subsets of any topological space form a (complete) distributive lattice, and so do the closed ones.

Moreover any sublattice or product of distributive lattices is again distributive. Hence so is any power D^X of any distributive lattice D with a partially ordered system X as exponent.† Also, the dual of any distributive lattice is distributive.

Therefore the lattice of the natural integers, ordered by divisibility, forms a distributive lattice: it is a sublattice of the direct product of a countable set of chains (consisting of the powers of the various primes 2, 3, 5, $\cdots$). Hence so does the (isomorphic‡) lattice of ideals of the ring of all algebraic integers of any finite extension of the rational field.

Many other examples of distributive lattices are described in Chapter VI. While in Chapter VII, it is shown that most *function spaces* are also distributive lattices.

90. Representation theory. It has been remarked that any ring of sets is a distributive lattice; in §§90–5 we shall prove converses of this proposition. These allow one to identify the theory of distributive lattices with that of rings of sets, classified relative to isomorphism.

In the finite case the argument is simple.§

THEOREM 5.3: *Every finite distributive lattice L is isomorphic with the ring of all M-closed subsets of the set X of its join-irreducible elements $x_i > O$.*

Proof: Clearly every a in L is the join $V_A x_i$ of the set A of join-irreducible elements $x_i \leqq a$; this is even true of O. Again, if $x_i \leqq a$ and $x_j \leqq x_i$, then $x_j \leqq a$; hence every set A is M-closed in X. We come now to the crux of the

† These conclusions apply to the family of lattices characterized by *any* identity or set of identities; the same considerations show that any lattice-homomorphic image of a distributive lattice is distributive. For the immense generality of this principle cf. G. Birkhoff [6].

‡ Cf. van der Waerden, vol. 2, p. 100. This is probably why Dedekind called distributive lattices "von Idealtypus."

§ The results and ideas of §90 and §93 are due to the author ([1], Thms. 17.2–17.3; also Comptes Rendus, 201 (1935), p. 19, and [7]); §93 involves ideas of Ore. The general ideas of representation theory for Boolean algebras and distributive lattices are due to Stone [4] and [6]. Stone was guided by the classical theories for the "representation" by matrices of finite groups, Lie algebras, and hypercomplex numbers.

proof: every M-closed set C in X contains all $x_j \leqq V_C x_i$ in X. Indeed, under our hypotheses,

$$x_j = x_j \frown \underset{C}{V} x_i = \underset{C}{V} (x_j \frown x_i)$$

and so (since x_j is join-irreducible) some $x_j \frown x_i$ is x_j, whence $x_j \leqq x_i$, and, C being M-closed in X, x_j is itself in C.

It follows that the correspondence between elements of L and M-closed subsets of X is one-one. But it clearly preserves order; hence it is an isomorphism, q.e.d.

Now applying Theorem 1.12, we infer immediately

COROLLARY 1: *L is B^Y, where Y is the dual of X.*

91. Hypothesis of finite dimensions. We shall next replace the hypothesis that L is finite by the (weaker) hypothesis that its dimensions are finite. Indeed,

LEMMA 1: *If L contains n join-irreducible elements $x_1, \cdots, x_n$, then $d[L] \geqq n$.*

Proof: Rearrange the x_i so that $x_i < x_j$ implies $i < j$; by the anti-circularity of partial ordering, this is possible. Then the chain $O < p_1 < p_1 \smile p_2 < \cdots < V_{i=1}^{n} p_i$ is of length n.

It is a corollary that if $d[L]$ is finite then the set X of Theorem 5.3 is finite, and so the remarks made above about M-closed subsets of X are true. Moreover the first two sentences of the proof of Theorem 5.3 apply to any lattice of finite dimensions—hence we can conclude that the entire proof, and so the theorem, does.

Lemma 1 shows that in Theorem 5.3 the dimensions of L are at least equal to the number of elements in X. But clearly the dimension of a ring of subsets of n elements is at most n; hence the two numbers are equal. We infer

THEOREM 5.4: *In Theorem 5.3, $d[L]$ equals the number of elements in X.*

COROLLARY 1: *The number of (non-isomorphic) distributive lattices of n dimensions is equal to the number of partially ordered systems of n elements.*

COROLLARY 2: *No distributive lattice of n dimensions contains more than 2^n elements.*

92. General representation theory. The "representation theory" for a distributive lattice L is concerned with the following problems: (I) with which rings of sets is L homomorphic, and (II) which of these homomorphisms are isomorphisms.

Accordingly, let θ be any homomorphism $x \to X$ from L to a ring $\mathfrak{R}$ of subsets $X, Y, Z, \cdots$ of a class I. Clearly each point $p \in I$ defines a partial representation θ_p of L onto a ring $\mathfrak{R}_p$ of subsets of p; moreover θ is in an obvious sense the sum of the θ_p.

But a given θ_p may be trivial: if every set in $\mathfrak{R}$ contains p, or if every set in $\mathfrak{R}$ excludes p, then θ_p will carry every element of L into the same element of $\mathfrak{R}_p$.

If θ_p is not trivial, it defines a homomorphism of L onto the two-element lattice B of all subsets of p, graphed in Fig. 11. Moreover the antecedents of O constitute an ideal J_p in L.

DEFINITION 5.2: *An ideal J in a lattice L will be called prime, if and only if J is the set of antecendents of O under some homomorphism of L onto the two-element lattice B.*

FIG. 11

Now if we discard those points p such that θ_p is trivial, and identify points p and q whenever $\theta_p = \theta_q$, we get an isomorphic "reduced" representation. In this, O and I in L correspond respectively to O and I in $\mathfrak{N}$, and each θ_p defines a different homomorphism $L \to B$. Moreover in virtue of our definition of prime ideals, we know

THEOREM 5.5: *The different "reduced" representations of L correspond one-one to the different subsets of the set of its prime ideals.*

Stone has defined the "perfect" representation of L as the "reduced" representation corresponding to the set of all its prime ideals. It is clear that, if L has *any* isomorphic representation, then its *perfect* representation will be isomorphic.

93. Digression on prime ideals. We shall now establish a connection between prime ideals and irreducible elements.

LEMMA 1: *An ideal J of a lattice L is prime if and only if $L - J$ is an ideal in the dual of L.*

Proof: That $L - J$ must be a prime ideal in the dual of L is evident: dualization interchanges J and $L - J$, and O and I in B. Conversely, if $L - J$ is an ideal in the dual of L, then (1) $x \in J$ and $y \in J$ imply $x \frown y$ and $x \smile y$ both in J, (2) $x \in L - J$ and $y \in L - J$ imply $x \frown y$ and $x \smile y$ in $L - J$, and (3) $x \in J$ and $y \in L - J$ imply $x \frown y$ in J and $x \smile y$ in $L - J$. Thus the partition of L into J into $L - J$ defines a homomorphism $L \to B$.

LEMMA 2: *J is prime if and only if $x \frown y \in J$ implies $x \in J$ or $y \in J$.*

Proof: This amounts to asserting that $x \notin J$ and $y \notin J$ imply $x \frown y \notin J$.

LEMMA 3: *A principal ideal $a \frown L$ of a distributive lattice L is prime, if and only if a is meet-irreducible.*

For $x \frown y \in a \frown L$ if and only if

$$a = a \smile (x \frown y) = (a \smile x) \frown (a \smile y),$$

and to say that this holds if and only if $x \in a \frown L$ or $y \in a \frown L$ is to say that it holds if and only if $a = a \smile x$ or $a = a \smile y$, that is, that $a = u \frown v$ if and only if $a = u$ or $a = v$.

Now let L be any distributive lattice of finite dimensions, so that every ideal in L or in its dual will be principal.

THEOREM 5.6: *The set X of meet-irreducible elements (ordered by relativization) is isomorphic with the set of prime ideals (ordered by set-inclusion). It is dually isomorphic with the set Y of join-irreducible elements.*

Proof: The first statement is a corollary of Lemma 3 above; the second follows since (1) prime ideals J in L and prime ideals $L - J$ in the dual of L are complementary in pairs, (2) complementation inverts set-inclusion, and (3) dualization carries meet- into join-irreducibility.

COROLLARY: *A distributive lattice L of finite dimensions n has exactly n prime ideals.*

Since a ring of subsets of a set of less than n elements has less than n dimensions, we conclude (by Theorem 5.5) that L has no isomorphic "reduced" representations, except the "perfect" representation.

94. Distributive lattices quâ modular lattices. If one assumes Theorems 3.5–3.6, many of the above results become much easier to prove.

Indeed, these theorems show that any modular lattice L of finite dimensions n is a sublattice of the product $L_1 \cdots L_r$ of "simple" homomorphic images $L_1, \cdots, L_r$, where $d[L_1] + \cdots + d[L_r] = d[L]$, and so $r \leqq n$.

But now if $r = n$, then $d[L_1] = \cdots = d[L_r] = 1$, and so every L_i is isomorphic with B, the only one-dimensional lattice. It follows that L is distributive. Conversely, if L is distributive, then from any connected chain $O = a_0 < a_1 < \cdots < a_n = I$ in L, one can construct n endomorphisms $\theta_i : x \to (x \smile a_{i-1}) \frown a_i$ of L, each of which defines a distinct homomorphism of L onto B—where B is obviously simple. We infer

THEOREM 5.7: *For a modular lattice L of finite dimensions to be distributive, each of the following conditions is necessary and sufficient:* (a) *the only "simple" homomorphic image of L be B,* (b) *L have $d[L]$ distinct prime congruence relations,* (c) *no two distinct quotients in the same chain be projective.*

COROLLARY: *A distributive lattice L of n dimensions has exactly n prime ideals.*

95. A general existence theorem. Much less is known in the infinite-

dimensional case. But even this case is contained in the following general representation theorem,

THEOREM 5.8:† *Any distributive lattice L is isomorphic with a ring of sets. (The converse is obvious.)*

Proof: We shall show that the perfect representation is isomorphic—that unless $x \geqq y$ in L, there exists a homomorphism $\theta: L \to B$ such that $\theta(x) = O$ and $\theta(y) = I$.

First make the homomorphism $\theta_1 : u \to (u \smile x) \frown y$, mapping L onto the sublattice $L_1 = L(x \frown y, y)$ of t satisfying $x \frown y \leqq t \leqq y$. Since $\theta_1(x) = x \frown y$ and $\theta_1(y) = y$, this reduces the problem to the case $x = O$, $y = I$.

Now use induction. If L_α contains an element z, $O < z < I$, form $L_{\alpha+1} = L_\alpha/z \frown L_\alpha$—i.e., identify elements congruent under the homomorphism $\theta_\alpha : u \to u \smile z$. Clearly $\theta_\alpha(O) < \theta_\alpha(I)$. If ω is a limit ordinal, define $x \equiv y \bmod \theta_\omega$ to mean $x \equiv y \bmod \theta_\alpha$ for some $\alpha < \omega$. That θ_ω *is* a lattice-homomorphism of L follows since the operations are binary;‡ moreover since $O \equiv I \bmod \theta_\alpha$ for no $\alpha < \omega$, $O \not\equiv I \bmod \theta_\omega$.

This process defines a (transfinite) sequence of congruence relations on L_1, each more inclusive than the last. Hence it cannot continue indefinitely, for the number of different congruence relations is bounded. But it continues until L contains no element besides O and I, and we have our desired homomorphism $L \to L_1 \to B$.

Remark: Applying Theorem 5.8 to L_1, we see that unless y covers $x \frown y$, there is more than one homomorphism $L \to B$ mapping x onto O and y onto I. Hence if we delete θ from the perfect representation, we get a second isomorphic "reduced" representation, unless θ is obtained from an endomorphism $u \to (u \smile x) \smile y$, where y covers $x \frown y$.

But now unless L is *connected*, the homomorphism θ^* which maps L onto the lattice of its connected subsystems (§56) annuls all prime quotients, and so is not of this type, nor is any homomorphism obtained from θ^* by further identification. Hence by our above remark, we conclude that if the only isomorphic "reduced" representation of a distributive lattice L is the perfect one, then L is connected. It is not hard to show that this condition is also sufficient.

96. Distributive functionals. We shall now obtain a characterization of those modular functionals which define *distributive* metric lattices.

This is easy; since $x \smile (y \frown z) \leqq (x \smile y) \frown (x \smile z)$ in any lattice,

$$m[x \smile (y \frown z)] = m[(x \smile y) \frown (x \smile z)]$$

† G. Birkhoff [1], Thm. 25.2. The proof is a simplification of a construction used first by Ulam (Fund. Math., 14 (1929), 231–3) and Tarski (ibid., 15 (1930), 42–50), and independently by several other authors. An interesting historical discussion of this construction and related questions has been made by M. H. Stone, *The representation of Boolean algebras*, Bull. Am. Math. Soc., 44 (1938), 807–16.

‡ If $x \equiv y \bmod \theta_\alpha$ and $x' \equiv y' \bmod \theta_\beta$, then $x \frown y \equiv x' \frown y'$ and $x \smile y \equiv x' \smile y' \bmod \theta_\gamma$, for all γ greater than α and β.

is necessary and sufficient for distributivity. But using condition M1,

$$m[x \smile (y \frown z)] = m[x] + m[y \frown z] - m[x \frown y \frown z]$$
$$= m[x] + m[y] + m[z] - m[y \smile z] - m[x \frown y \frown z],$$
$$m[(x \smile y) \frown (x \smile z)] = m[x \smile y] + m[x \smile z] - m[x \smile y \smile z].$$

Substituting and transposing, we get the equivalent *symmetric* condition (writing $x = x_1$, $y = x_2$, $z = x_3$)

$$m[x_1 \smile x_2 \smile x_3] - m[x_1 \frown x_2 \frown x_3] = \sum_{i \neq j} m[x_i \smile x_j] - \sum_i m[x_i].$$

But this condition is not self-dual; however, we know by M2 that

$$\sum_{i \neq j} m[x_i \smile x_j] - \sum_i m[x_i] = \sum_i m[x_i] - \sum_{i \neq j} m[x_i \frown x_j];$$

hence it is equivalent to the *self-dual, symmetric* condition

$$\text{M3: } 2\{m[x_1 \smile x_2 \smile x_3] - m[x_1 \frown x_2 \frown x_3]\} = \sum_{i \neq j} \{m[x_i \smile x_j] - m[x_i \frown x_j]\}.$$

The existence of such a symmetric, self-dual condition might of course be inferred from L6.

We shall call a modular functional "distributive," if and only if it satisfies M3.

97. Neutral elements. One can sharpen Theorem 5.1, by observing that in §62, the quotients c_i/b_i, a_i/d_i, e/c_i, and e_i/d are projective. Hence if one of them is annulled, they all are, and so is e/d. We conclude

THEOREM 5.9: *Let x, y, z be elements of a modular lattice. If one of the equalities L6, L6′, L6″ holds, then the sublattice generated by x, y, z is distributive.*† *The second example of Fig. 10 shows that the assumption of modularity can not be omitted.*

It is a corollary that an element a of a modular lattice is neutral if and only if the correspondences $x \to a \smile x$ and $x \to a \frown x$ are *both* lattice endomorphisms. But now if a and b are both neutral, then the product $x \to x \smile (a \smile b)$ of the endomorphisms $x \to x \smile a$ and $y \to y \smile b$ is itself an endomorphism. Hence $a \smile b$ is neutral; dually, $a \frown b$ is neutral, and so

THEOREM 5.10: *The neutral elements of any modular lattice L constitute a (distributive) sublattice of L.*

Incidentally, it is easy to verify that the neutral elements of any product of lattices are the elements all of whose components are neutral, also, that the center of a modular lattice consists of its complemented neutral elements.

98. Note on projectivity of quotients. We have seen (§72) that it is possible for a quotient in a modular lattice to be projective with a part of itself; we shall now see that this is impossible in a distributive lattice.

† Ore [1], p. 416, Thm. 3. Theorem 5.4 on neutral elements is also due to Ore—ibid., p. 421.

To see this, consider the system Σ of endomorphisms of a lattice L—and more particularly, the subsystem Σ^* of Σ generated by endomorphisms of the forms $x \to x \smile a$ and $x \to x \frown a$. Since the transformation $x \to x \smile a$ carries $x/x \frown a$ into $x \smile a/a$ and dually, we see that any transposition can be accomplished by members of Σ^*—and hence that so can any projectivity.

Again, since $(x \smile a) \smile b = x \smile (a \smile b)$ and dually, we see that any member of Σ^* can be put into the form

$$x \to [(x \smile a) \frown b] \smile c \cdots$$

or its dual. While in a distributive lattice, since

$$(x \smile a) \frown b = (x \frown b) \smile (a \frown b) = (x \frown b) \frown c,$$

one can transpose cups and caps—and hence rewrite any transformation of Σ^* both in the form $x \to (x \smile a) \frown b$ and in a dual form.

Now suppose x/y and x_1/y_1 are projective, where $x \leqq x_1$ and $y \geqq y_1$. Then, writing $x_1 = (x \smile a) \frown b$ and $y_1 = (y \smile a) \frown b$, we will have $x \leqq b$ and so $y_1 \geqq (y \smile a) \frown x \geqq y \frown x = y$, proving $y = y_1$. Dually, $x = x_1$, which proves

THEOREM 5.11: *In a distributive lattice, no quotient x_1/y_1 is projective with a proper part of itself.*

99. Uniqueness of decomposition of elements. Let L be any (finite or infinite) distributive lattice, and let

$$x = x_1 \frown \cdots \frown x_r = y_1 \frown \cdots \frown y_s$$

be any two representations of an element of L as a meet. Then for any i,

$$x_i = x_i \smile x = x_i \smile (\bigwedge_j y_j) = \bigwedge_{j=1}^{s} (x_i \smile y_j).$$

Hence if x_i is meet-irreducible, some $x_i \smile y_j = x_i$—whence some $y_j \leqq x_i$. Similarly, if y_j is meet-irreducible, some $x_k \leqq y_j$. Hence either $x_k = y_j = x_i$, or x_i is redundant in the strong sense that some $x_k < x_i$, whence

$$x = x_1 \frown \cdots \frown x_{i-1} \frown x_{i+1} \frown \cdots \frown x_r.$$

Thus if the decompositions are irredundant, then the x_i and y_j are equal in pairs, $r = s$, and we conclude

LEMMA 1: *In a distributive lattice, no element has more than one irredundant meet-decomposition into meet-irreducible elements (and dually).*

Conversely, a modular lattice which is not distributive contains by Theorem 5.2 the sublattice of Fig. 10, §88, as a sublattice. Now starting with the two meet-decompositions $o = x \frown u$ and $o = x \frown v$, decomposing further as long as possible, and finally eliminating redundant components, we see that any factor for the second decomposition which contains u must contain x or v and hence e,

whereas the first decomposition and those derived from it possess at least one factor containing u but not e. Hence if the above process is terminating, we will get two distinct irredundant decompositions of a into meet-irreducible elements. But in the presence of the ascending chain condition, the process is terminating, and so

THEOREM 5.12: *A modular lattice satisfying the ascending chain condition is distributive, if and only if each element has a unique irredundant decomposition into meet-irreducible elements.*†

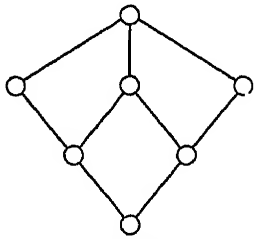

FIG. 12

The example graphed in Fig. 12 shows that the hypothesis of modularity cannot be omitted. The dual of Theorem 5.12 is obviously also true, by dual reasoning.

100. General finite distributivity. In L6′, L6″ we recognize two dual forms of the distributive law, involving three variables. We shall now generalize these to distributive laws involving n variables.

To express such laws, we shall want the notation $\Lambda_{i=1}^{m} u_i$ for $u_1 \frown \cdots \frown u_m$ and $V_{i=1}^{m} u_i$ for $u_1 \smile \cdots \smile u_m$. It then follows by induction on n that

$$x \frown \mathop{V}_{i=1}^{n} y_i = \mathop{V}_{i=1}^{n} (x \frown y_i) \qquad (1) \quad \text{and dually.}$$

Now using induction on m, we obtain

$$\mathop{V}_{i=1}^{m} x_i \frown \mathop{V}_{j=1}^{n} y_j = \mathop{V}_{i,j} (x_i \frown y_j) \qquad (2) \quad \text{and dually.}$$

Finally, using induction on the number r of terms, we can derive the *generalized* distributive law

$$\mathop{\Lambda}_{h=1}^{r} \left[\mathop{V}_{k=1}^{n(h)} u_{h,k} \right] = \mathop{V}_{f} \left[\mathop{\Lambda}_{h=1}^{r} u_{h,f(h)} \right] \qquad (3) \quad \text{and dually,}$$

where f runs through all functions $k = f(h)$ which associate with each h a unique

† This result was announced by the author in Abstract 41-1-75, Bull. Am. Math. Soc., 41 (1935), p. 32. A necessary and sufficient condition that the elements of a general lattice have unique such decompositions has been obtained by R. P. Dilworth, *Lattices with unique irreducible decompositions*, Bull. Am. Math. Soc., 45 (1939), p. 833.

$k = 1, \cdots, n(h)$. This practically contains L6 as a special case, with $r = 3$ and $n(1) = n(2) = n(3) = 2$.

THEOREM 5.13: *Any lattice polynomial ϕ of elements $x_1, \cdots, x_m$ of a distributive lattice L can be written in the form*

$$\bigwedge_{i=1}^{r} \left[\bigvee_{k=1}^{n(h)} x_{i(h,k)} \right],$$

and also dually.†

Proof: By repeated use of the generalized distributive law, we can replace $\wedge V$ by $V\wedge$ and conversely. Hence we can replace any $\wedge V \wedge V \cdots$ by $\wedge\wedge VV \cdots$, and this (using the associative law) by $\wedge V \cdots$. The conclusion is now obvious by induction.

101. Free distributive lattice. We shall now determine the free distributive lattice generated by n symbols $x_1, \cdots, x_n$, having a O and a I adjoined.

In the first place, using Greek letters to denote sets of x_i, and Roman capitals to denote families of such sets, we can write every element in the form $V_{\sigma \in F} (\wedge_{i \in \sigma} x_i)$. This is a corollary of Theorem 5.13, since a meet of x_i is (by L1–L3) determined by the *set* σ of x_i involved. Again, if $\sigma^* \geqq \sigma$, then

$$\bigwedge_{i \in \sigma} x_i \cup \bigwedge_{i \in \sigma^*} x_i = \bigwedge_{i \in \sigma} x_i$$

—hence every element can be written in the form $V_{\sigma \in F^*} (\wedge_{i \in \sigma} x_i)$, where F^* is the J-*closure* of F.

Now we shall go back, and show that different F^* correspond to different elements, thus obtaining a one-one correspondence between F^* and elements generated by the x_i. Indeed, let I be the family of all sets σ of x_i —and let X_i be the family which contains σ if and only if $x_i \in \sigma$. Then X_i will be J-closed, and the ring of sets which the X_i generate will include only J-closed sets. Moreover $\wedge_{i \in \tau} X_i$ will contain σ if and only if every x_i [$i \in \tau$] is in σ —i.e., if and only if $\sigma \geqq \tau$. Hence $V_{\tau \in G} (\wedge_{i \in \tau} X_i)$ contains σ if and only if $\sigma \geqq \tau$ for some $\tau \in G$ —hence, if G is J-closed, if and only if $\sigma \in G$. Hence different G determine different $V_G (\wedge_\sigma X_i)$, q.e.d.

But the one-one correspondence set up is order-preserving, by our last conclusion; hence it is isomorphic. Moreover we get every J-closed subset of I, with two exceptions: every subset of σ which we obtain contains I, and none contains O. Hence if we adjoin a O (the void set, not containing I), and a I containing even O as a member), we get

THEOREM 5.14: *The free distributive lattice generated by n symbols, and having a O and I is isomorphic with the ring of J-closed subsets of the field B^n of all subsets of n points.*‡

† This principle, together with formulas (1)–(3), has been known almost since Boole; cf. Boole [2], pp. 72–5. Cf. also the author [1], Thm. 16.3.

‡ This result is essentially due to Th. Skolem [2].

COROLLARY 1: *The free distributive lattice generated by n symbols, and having a O and I, is B^{B^n}.*

We note that the free modular lattice generated by two chains of lengths m and n is by Theorem 3.18 distributive, and (letting C_k denote a chain of k elements) is $B^{C_m C_n}$.

COROLLARY 2: *The sublattice generated by a given finite subset of any distributive lattice is finite.*

We note that although B^{B^n} has 2^n dimensions, the function $f(n)$ expressing the number of elements in B^{B^n} is yet to be described.*

102. Infinite distributivity. Formulas (1)–(3) of §100 hold automatically for finite n. This brings up the question, do they hold for *any* cardinal number, in *any* distributive lattice? They do in any complete ring of sets.

LEMMA 1: *Identity* (1) *holds in a distributive lattice for n countable, if and only if it is a topological lattice.*

Proof: Set $s_n = y_1 \smile \cdots \smile y_n$ and $s = V_{i=1}^{\infty} y_i$. Then $s_n \uparrow s$, $\{x \frown s_n\} \uparrow$, $x \frown V_{i=1}^{\infty} y_i = x \frown s$, and—since

$$\mathop{V}_{i=1}^{n} (x \frown y_i) = x \frown \mathop{V}_{i=1}^{n} y_i = x \frown s_n$$

—$V_{i=1}^{\infty}(x \frown y_i)$ is sup $x \frown s_n$. Hence to assert (1) is to assert that $s_n \uparrow s$ implies $x \frown s_n \uparrow x \frown s$, and dually. But this is the same as asserting that the lattice is topological.

Remark: The (perhaps less interesting) *uncountable* case can be handled in the same fashion, if one uses generalized limits.

LEMMA 2: *Identities* (1) *and* (2) *of* §100 *are equivalent in any complete lattice — and for countable m, n in any σ-lattice.*†

Proof: The implication (2)–(1) is trivial; set $m = 1$. But using (1) twice,

$$\mathop{V}_{i} x_i \frown \mathop{V}_{j} y_j = \mathop{V}_{i} (x_i \frown \mathop{V}_{j} y_j) = \mathop{V}_{i}(\mathop{V}_{j} x_i \frown y_j),$$

and by the generalized associative law, this is $V_{i,j}(x_i \frown y_j)$. Similarly for the dual of this.

We shall see in Chapter VI that (3) is by no means a consequence of its special case (2).

103. Application to abstract algebra. We know by Theorem 3.6 that the congruence relations on any modular lattice of finite dimensions form a distributive lattice. The same is true of semi-simple groups and semi-simple hyper-

* Dedekind [1], p. 147 proposed this problem. We know $f(1) = 3$, $f(2) = 6$, $f(3) = 20$, $f(4) = 168$. This suggests that $f(n)$ is always a multiple of $(2n - 1)(2n - 2)$, if $n > 1$.

† Lemmas 1–2 are due to von Neumann [2], Appendix 1 of Part III.

complex algebras. It is also true of integrally closed rings of algebraic numbers —but not (for example) of polynomial rings.

This distinction is fundamental; we get, as immediate corollaries of Theorems 5.12 and 3.20,

THEOREM 5.15: *Let A be any abstract algebra, whose congruence relations satisfy the ascending chain condition. Then A and its homomorphic images have unique representations as subdirect unions of irreducible factors, if the lattice of its congruence relations is distributive. If A is a group with or without operators, the condition is sufficient as well as necessary, and A has a strictly unique direct decomposition.*

Theorem 5.2 and its second corollary give one elegant criterion for deciding whether the homomorphisms of a group G (with or without operators) form a distributive lattice.† It shows, applying Theorem 3.16, that unless this is the case, G has a homomorphic image G_θ with two "independent" operator-isomorphic Ω-subgroups M and N.

We note that if G is a group, then since the elements of M are left invariant under the automorphisms of N, so are those of N, and so N is Abelian. Similarly, since multiplication by elements of N carries elements of M into O, it carries elements of N into O, and so $NN = O$.

This is related to the classical antithesis between nilpotence and semi-simplicity which is so basic in the theories of Lie algebras and associative algebras of Cartan and of Molien-Peirce-Wedderburn.

104. Applications to topology: bicompact spaces. Let I be any T_1-space—that is,‡ any aggregate of "points," each set S of which has a "topological closure" $\bar{S}$, where

C1: $S \leqq \bar{S}$. C2: $\bar{\bar{S}} = \bar{S}$.
C3*: $\overline{S + T} = \bar{S} + \bar{T}$. C4: *If p is a point, then* $p = \bar{p}$.

The operation $S \to \bar{S}$ is a "closure operation" in the sense of §19 (C3* implies C3), and so by Theorem 2.2 the closed subsets of I form a *complete lattice* $L(I)$ — which is distributive since the intersection and (by C3*) the sum of any two closed sets is closed.

It is obvious that I is characterized topologically by $L(I)$. For I consists by C4 of the elements which cover O in $L(I)$, and the closure of any set $S \leqq I$ is the set $\bar{S}$ of points contained in the lattice-join of the $p \in S$.

In fact, it is easy to show that an abstract complete distributive lattice is the

† The second criterion is applicable in a surprising number of cases, in virtue of a celebrated theorem of Hilbert (cf. van der Waerden [1], vol. 2, pp. 23–7).

‡ In the sense of Alexandroff-Hopf, *Topologie*, Berlin, 1935, p. 59. This is also Kuratowski's definition of a topological space (*Topologie*, Warsaw, 1933). The results given below are due to Wallman [1], [2]; they were suggested by, and partly generalize, Stone's application of Boolean algebra to totally disconnected bicompact T_1-spaces (cf. §118).

lattice of all closed subsets of a T_1-space, if and only if, given $S > T$, there exists a p covering O such that $p \leqq S$ yet $p \frown T = O$.

Now define a "basis" as a sublattice B of $L(I)$, such that every member of $L(I)$ is a meet of members of B; clearly the topology of I is determined by any basis of closed sets.† It is clear that the subsets of B containing any point p constitute a maximal dual ideal J_p —and that $p \epsilon S$ [$S \epsilon B$] if and only if $S \epsilon J_p$.

But one cannot tell from B alone which of its maximal dual ideals correspond to points of I. This is true of all maximal dual ideals, if and only if I is bicompact.‡ Hence a *bicompact* space is characterized topologically by *any* basis of closed sets. Wallman has shown that an abstract distributive lattice with O and I is isomorphic to a basis of closed sets of a bicompact T_1-space, if and only if it has the "disjunction property": given $S > T$, there exists an X such that $S \frown X > O$ yet $T \frown X = O$.

Furthermore, if we define I^* as the set of maximal dual ideals J of $L(I)$, associate with each $S \epsilon L(I)$ the set $S^* = I^*$ of J such that $S \epsilon J$, and regard the S^* as forming a *basis* of closed subsets of I^*, we get a bicompact space of which I is a dense subset; the proof is immediate (cf. Wallman [1]).

Incidentally, I^* has the same dimension and the same homology theory as I, in the sense of Cech.

One can observe further that many bicompact spaces have countable bases of closed sets—and that any countable basis can be expressed as the limit of a monotone sequence of finite sublattices. This remark suggests Alexandroff's characterization of separable bicompact spaces by sequences of "finite coverings" (abstract complexes), each a refinement of the last.§

† A set being closed if and only if it is the intersection of subsets of the basis. Dually, we have Hausdorff's notion of a "neighborhood system," such that every open set is the sum of neighborhoods of the system.

‡ For a maximal dual ideal is a maximal class of closed subsets, no finite subfamily of which has a void intersection. That the intersection of such a class of closed subsets should be a point is one definition of bicompacticity.

§ Further literature: H. Terasaka, *Über die Darstellung der Verbände*, Proc. Imp. Acad. Japan, 14 (1938), 306–11; J. W. Alexander, *A theory of connectivity for gratings*, Annals of Math., 39 (1938), 883–912, and Princeton Lectures (1940); A. N. Milgram, Reports of Notre Dame Colloquium (1940).

CHAPTER VI

BOOLEAN ALGEBRAS

105. Definition. Historically, lattice theory began with Boolean algebra,* in the sense of

DEFINITION 6.1: *A Boolean algebra is a complemented distributive lattice.*

For conditions equivalent to complementedness, cf. Theorem 4.1.

The close connection between the distributive law and unicity of complementation has already been pointed out (cf. Theorem 4.9 and Corollary 1 of Theorem 5.2). In fact, one can prove

THEOREM 6.1: *In a distributive lattice, complementation is unique and is orthocomplementation.*†

Proof: If $a \smile x = I$ and $a \frown y = O$, then

$$x = O \smile x = (a \frown y) \smile x = (a \smile x) \frown (y \smile x) = I \frown (y \smile x) = y \smile x.$$

If also $a \frown x = O$ and $a \smile y = I$, then similarly $y = y \smile x$, whence $x = y$, proving unicity.

But by L2, the relation of complementarity is symmetric, and so $(a')' = a$. Again, if $a \leqq b$, then $a \smile a' = I$ and $a \frown b' \leqq b \frown b' = O$. Hence (as above) $a' = b' \smile a'$, and $b' \leqq a'$. That is, the correspondence $a \rightarrow a'$ is a dual automorphism, completing the proof.

From this theorem it follows directly that any Boolean algebra is dually isomorphic with itself. It is a second corollary that one can axiomatize Boolean algebra in terms of meets and complements alone; this has been done in many ways in the literature on the subject.‡ It is a third corollary that any lattice-automorphism of a Boolean algebra preserves complementation: the operation of complementation is *intrinsic.*

THEOREM 6.2: *The complemented elements of any distributive lattice form a sublattice.*

Proof: If x and y are complemented, then we have $(x \frown y) \frown (x' \smile y') =$

* Of course, Boolean algebra goes back to Boole [1]. Even in 1897 A. N. Whitehead wrote (*Universal Algebra*, p. 35) that "Boolean algebra is the only known member of the non-numerical genus of universal algebra."

† The result goes back to R. Grassmann; cf. Schröder [1], pp. 299, 305, 352 for unicity, $(x')' = x$ and L7″ respectively.

‡ For a brilliant axiomatization of Boolean algebra in terms of a single operation cf. H. M. Sheffer, *A set of five independent postulates for Boolean algebras*, Trans. Am. Math. Soc., 14 (1913), 481–8.

$(x \frown y \frown x') \smile (x \frown y \frown y') = O$ and dually; hence $x \frown y$ has $x' \smile y'$ for complement. Dually, $x \smile y$ is complemented—and in fact, $x' \frown y' = (x \smile y)'$.

106. Examples. The subsets of any aggregate I form a Boolean algebra: the set-theoretical notions of sum, product and complement become the lattice-theoretical notions of join, meet and complement, respectively.

More generally, any field of sets is a Boolean algebra, with the same interpretations.

Again, consider the different (dyadic) relations between the elements of two classes I and J. And suppose we identify each relation ρ with the set $X(\rho)$ of pairs of elements in that relation. Then $X(\rho) \leqq X(\rho')$ if and only if all elements in the relation ρ are in the relation ρ'; that is, if and only if ρ *implies* ρ'. Hence we can interpret the different relations between the elements of I and those of J as a Boolean algebra.

Any product of Boolean algebras is itself a Boolean algebra; so is any "subalgebra" of a Boolean algebra—i.e., any subset closed with respect to all three Boolean operations. Also, any lattice-homomorphic image of a Boolean algebra is itself a Boolean algebra: it is a distributive lattice, and the equations $x \frown x' = O$, $x \smile x' = I$ are preserved.

THEOREM 6.3: *The center of any lattice with O and I is a Boolean algebra.*

Proof: It is a sublattice; all its elements are neutral; all its elements have complements in the center.

It is a corollary that the neutral elements of any complemented modular lattice form a Boolean algebra.

Other examples of Boolean algebras will be discussed below in Chapter VIII.

107. Representation theory. The theory of the representation of Boolean algebras by fields of sets can be obtained by specialization from the theory of the representation of distributive lattices by rings of sets, given in §§90–5.

In the first place, "reduced" representations of any Boolean algebra by rings of sets are by fields of sets (since $O \rightarrow O$ and $I \rightarrow I$). Hence Theorem 5.5 applies, and we get besides, as a corollary of Theorem 5.8, Stone's result:

THEOREM 6.4: *Any Boolean algebra is isomorphic with a field of sets. (The converse is obvious.)*

Again, consider the finite-dimensional case. We know (§65) that an element a of a complemented modular lattice of finite dimensions is join-irreducible if and only if it covers O—for if $O < x < a$, then $a = x \smile (x' \frown a)$. Hence the X and Y of Theorem 5.6 are unordered aggregates, and

THEOREM 6.5: *Every Boolean algebra of finite dimensions is isomorphic with the field of all subsets of an (unordered) aggregate of n elements.*

Thus in particular, there is just one Boolean algebra of each finite dimension n; it contains 2^n elements.

108. Congruence relations. The theory of the congruence relations on a Boolean algebra A is dominated by two considerations.

Firstly, since A is a complemented modular lattice, any congruence relation on it is determined by the set of elements congruent to O (Theorem 4.3). And secondly,†

THEOREM 6.6: *The subsets of A congruent to O under its different congruence relations are its different* ideals J.

Proof: We know that every congruence module is an ideal; we need to prove that any ideal J is a congruence module. But indeed, given J, if we make the definition that $x \equiv y\ (J)$ means $x \smile t = u \smile y$ for some $t, u \in J$, then we evidently have a reflexive and symmetric relation. It is also transitive, for if $x \smile t = u \smile y$ and $y \smile v = w \smile z$, then $x \smile (t \smile v) = u \smile y \smile v = (u \smile w) \smile z$. Hence it is an equivalence relation. Again, suppose $x \equiv y\ (J)$ and $x_1 \equiv y_1\ (J)$—that is, $x \smile t = u \smile y$ and $x_1 \smile t_1 = u_1 \smile y_1$ $[t, u, t_1, u_1 \in J]$. Then by L1–L4

$$x \smile x_1 \smile (t \smile t_1) = (x \smile t) \smile (x_1 \smile t_1) = (y \smile u) \smile (y_1 \smile u_1)$$
$$= y \smile y_1 \smile (u \smile u_1)$$

so that $x \smile x_1 \equiv y \smile y_1\ (J)$. Also, if we set $t \smile t_1 \smile u \smile u_1 = v$, then $v \in J$ and moreover

$$(x \frown x_1) \smile v = (x \smile v) \frown (x_1 \smile v) = (y \smile v) \frown (y_1 \smile v) = (y \frown y_1) \smile v$$

since if $x \smile t = u \smile y$, then $x \smile w = w \smile y$ for all $w \geqq t, u$. That is, $x \frown x_1 \equiv y \frown y_1\ (J)$, and we have defined a congruence relation. Finally, $x \equiv 0\ (J)$ if and only if $x \leqq x \smile t = u$ for some $u \in J$, which is to say, if and only if $x \in J$, so that J is the congruence module for this relation, q.e.d.

Incidentally, we note that any lattice-homomorphism carries complements into complements. Hence $x \equiv y\ (J)$ implies $x' \equiv y'\ (J)$, and lattice-homomorphisms are homomorphic with respect to all three Boolean operations.

It is a corollary of Theorem 6.6 that the lattice of congruence relations on any Boolean algebra is isomorphic with the (distributive) lattice of its ideals.

109. Calculus of ideals. Now consider the set $\mathfrak{A}$ of ideals of a Boolean algebra A, partially ordered by the relation of set-inclusion.

In the first place, the set-product of two ideals J and K is clearly an ideal, and is (in terms of the so-called "calculus of complexes")‡ the set $J \frown K$ of $s \frown t\ [s \in J, t \in K]$. Dually, any ideal which contains J and K contains all $s \smile t\ [s \in J, t \in K]$; conversely, the set $J \smile K$ of such joins contains with $s \smile t$ and $s^* \smile t^*$ also (1) $(s \smile t) \smile (s^* \smile t^*) = (s \smile s^*) \smile (t \smile t^*)\ [s \smile s^* \in J,$

† Theorem 4.4 might lead us to conjecture this result as it shows it is true in the case of principal ideals. Theorem 6.6 is due to Stone.

‡ Introduced by Frobenius, for groups. Indeed, $s \frown t \leqq s \in J$ and $s \frown t \leqq t \in K$, whence $s \frown t$ is in both J and K. Conversely, any u in both J and K can be written $u \frown u\ [u \in J, u \in K]$.

$t \smile t^* \in K$], and (2) all $u \leqq s \smile t$, since any such u can be written $u \frown (s \smile t) = (u \frown s) \smile (u \frown t)$ [$u \frown s \in J$, $u \frown t \in K$].

In summary, joins and meets of ideals are given by the calculus of complexes. The resulting calculus of ideals has been discussed exhaustively by Moisil [1] and Stone. It suggests the identities

$$J \smile (K \frown L) = (J \smile K) \frown (J \smile L),\ J \frown (K \smile L) = (J \frown K) \smile (J \frown L)$$

—that is, the fact that $\mathfrak{A}$ is a distributive lattice.†

Indeed, $J \frown (K \smile L) \leqq (J \frown K) \smile (J \frown L)$ set-theoretically, by the calculus of complexes, while the reverse inequality holds by Corollary 1 of Theorem 2.7.

110. Classification of ideals. As Stone has pointed out, the ideals of infinite Boolean algebras fall into several categories—notably, principal, normal, and prime ideals. (Cf. also Tarski, *Ideale in den Mengenkörpern*, Ann. Soc. Pol. Math., 15 (1937), 186–9.)

We have already defined principal ideals and prime ideals; now recall the definition of a "normal subset" (§33), as a set which includes all lower bounds to the set of its upper bounds. Clearly, any normal subset is an ideal; we distinguish those ideals which are normal subsets by calling them "normal ideals."

In §33, we observed in effect that every principal ideal was normal; we now observe that the converse holds in *complete* Boolean algebras and no others. Indeed, MacNeille's process of completing by cuts yields no new elements if and only if every normal subset is already a principal ideal.

Again, consider *prime* ideals P. In a Boolean algebra, unless $x \in P$, $x' \in P$; hence any ideal $J > P$ contains, with x [x not in P], $x \smile x' = I$—and hence is the entire Boolean algebra. Conversely, the congruence relation defined by any maximal ideal gives an image-lattice of only two elements. We conclude: an ideal of a Boolean algebra is prime if and only if it is *maximal*.‡

We shall conclude by proving a result of Stone: if all ideals of a Boolean algebra A are principal ideals, then A is finite; the converse is obvious.

Indeed, form any connected chain of elements between O and I in A. If this chain is finite, then by the Jordan-Dedekind condition A is finite-dimensional and so finite. If it is infinite, it contains an infinite ascending sequence or an infinite descending sequence; since complementation interchanges ascending and descending sequences, A contains *some* sequence $a_1 < a_2 < a_3 < \cdots$. Let J consist of the x such that $x \leqq a_k$ for some k. Clearly J is an ideal which has no greatest element—i.e., is not principal.

111. Subalgebras in finite-dimensional case. Consider again the finite-dimensional case, and let B^n denote the Boolean algebra of all subsets of a set

† It is a plausible conjecture that the calculus of complexes preserves all identities. But this is true only of laws which involve no letter twice on the same side—e.g., it is true of the commutative and associative, but not of the idempotent or distributive laws.

‡ As Stone has pointed out, this is the same situation that one has with ideals in the theory of algebraic numbers, where the quotient-ring over an ideal is a field if and only if the ideal is maximal.

I of n points. We know that the congruence relations on B^n are obtained by ignoring different subsets of I. But what are the *subalgebras* of B^n —what are the sublattices of B^n which contain with any x, also x', and therefore O and I.

Any subalgebra S will be a finite-dimensional Boolean algebra, whose "points" will be independent elements of B^n (disjoint subsets of I) whose join is I—that is, they will be the subsets into which some *partition* divides I. Conversely, the subsets into which any partition divides I are the indivisible or "atomic" members of a field of sets, so that the correspondence between subalgebras of B^n and partitions of I is one-one. Finally, $S \leqq T$ if and only if its "points" are joins of "points" of T, that is, if and only if T effects a subpartition of the partition effected by S. In summary,*

THEOREM 6.7: *The lattice of the subalgebras of the Boolean algebra of all subsets of any finite set I is dually isomorphic with the lattice of partitions of I.*

112. Free Boolean algebras. We shall now determine the free Boolean algebra generated by n symbols. This question can be restated as follows. By a "Boolean function" of variables $x_1, \cdots, x_n$ is meant one built up from the three basic functions: join, meet and complement. We ask: what are the different Boolean functions of the x_i, and how does one combine them? The answer was given by Boole himself ([1], pp. 72–5).

To get the answer, first form the 2^n "elementary" Boolean functions† f_i: $x_{i1} \frown \cdots \frown x_{in}$—where x_{ij} is either x_j or x_j', depending on i. We next note that distinct f_i are independent: $f_i \frown f_k \leqq x_{ij} \frown x_{kj}$ for all j, and if $i \neq k$, then $x_{ij} \frown x_{kj}$ is O (being $x_j \frown x_j'$ or $x_j' \frown x_j$), for at least one j.

Next associate with each non-void set S of f_i, the function $g_S = V_{i \epsilon S} f_i$, and define g_O as O. Observe: (1) $g_O = O$ and $g_I = I$, (2) $g_{S \smile T} = g_S \smile g_T$, $g_{S \frown T} = g_S \frown g_T$, and $g_{S'} = (g_S)'$, (3) every Boolean function of the x_i is a g_S.

Proof of (1): By definition, $g_O = O$, while by the general distributive law,

$$I = \bigwedge_{j=1}^{n} (x_j \smile x_j') = \bigvee_{i=1}^{2^n} \bigwedge_{j=1}^{n} x_{ij} = \bigvee_{i \epsilon I} f_i .$$

Proof of (2): $g_S \smile g_T = g_{S \smile T}$ by definition and L1–L3. Again, $g_S \frown g_T$ is the join of the $f_i \frown f_k$ $[i \epsilon S, k \epsilon T]$ by the general distributive law, while $f_i \frown f_k$ is O unless $i = k$. Hence $g_S \frown g_T = Vf_i \frown f_i$ $[i \epsilon S, i \epsilon T]$, which by L1 is Vf_i $[i \epsilon S \frown T]$, or $g_{S \frown T}$, q.e.d. It follows that $g_S \smile g_{S'} = I$ and $g_S \frown g_{S'} = O$, whence $g_{S'} = (g_S)'$.

Proof of (3): By (2), every Boolean function of a g_S is itself a g_S; hence it suffices to show that the x_i are g_S—and by symmetry, that x_1 is. But by the general distributive law, if X_1 denotes the set of i such that $x_{i1} = x_1$,

$$x_1 = x_1 \frown \bigwedge_{j=2}^{n} (x_j \smile x_j') = \bigvee_{i \epsilon X_1} f_i .$$

* This result is due to the author [6], Thm. 21.

† E.g., if $n = 2$, we have $f_1 = x_1 \frown x_2$, $f_2 = x_1 \frown x_2'$, $f_3 = x_1' \frown x_2$, $f_4 = x_1' \frown x_2'$.

It follows that every Boolean function is a g_S; we shall now show that distinct g_S represent distinct functions. Indeed, let I consist of all points with n coordinates, each 0 or 1 (the vertices of an n-dimensional cube). Denote by x_j the set of points with jth coordinate 1. Then each f_i will represent a different point: the point with jth coordinate 0 or 1 according as x_{ij} is x_j or x_j'. Hence distinct sets S determine distinct g_S in this case, and a fortiori in a free Boolean algebra.

Thus the elements of the free Boolean algebra generated by $x_1, \cdots, x_n$ are the different g_S. But (2) shows that these combine isomorphically with the subsets of a class I of 2^n elements, and so

THEOREM 6.8: *The free Boolean algebra generated by n symbols is isomorphic with the algebra B^{2^n} of all subsets of a space of 2^n points.*

113. Infinite distributivity. We distinguished in §§100–102 between two grades of infinite distributivity. The first is equivalent to continuity of the lattice operations in the intrinsic lattice topology; it is the condition that

(1) $x \frown V_\beta y_\beta = V_\beta x \frown y_\beta$ and dually.

It implies

(2) $V_\alpha x_\alpha \frown V_\beta y_\beta = V_{\alpha,\beta} x_\alpha \frown y_\beta$ and dually.

As has been observed by von Neumann ([2], II, Appendix, p. 7), with complemented lattices, finite distributivity alone implies (1) and (2). That is

THEOREM 6.9: *Identities* (1)–(2) *hold in a Boolean algebra whenever their terms are defined. Thus they hold identically in any complete Boolean algebra.*

Proof: By §102, Lemma 2, we need only prove (1), and by duality, only the first part of (1). Again, $x \frown V_\beta y_\beta$ is evidently an upper bound to every $x \frown y_\beta$; hence we need only show that $x \frown V_\beta y_\beta$ is contained in every upper bound a to the $x \frown y_\beta$. But if $x \frown y_\beta \leqq a$ for all β, then

$$y_\beta = (y_\beta \frown x) \smile (y_\beta \frown x') \leqq a \smile x' \qquad \text{for all } \beta$$

and so

$$x \frown V_\beta y_\beta \leqq x \frown (a \smile x') = (x \frown a) \smile (x \frown x') = x \frown a \leqq a.$$

114. Complete distributivity. The construction of §111, extended transfinitely, also yields the following rather surprising result of Tarski ([1], pp. 195–7),

THEOREM 6.10: *If a complete Boolean algebra A is completely distributive, then it is isomorphic with the algebra $B^{\aleph}$ of all subsets of some aggregate.*

Explanation: A lattice will be called "completely distributive," if and only if identity (3) of §100 holds without restriction on the number of terms involved.

Proof: Let x_α denote the general element of A; make the expansion $I = \Lambda_\alpha (x_\alpha \smile x_\alpha') = Vf_\phi$, where f_ϕ denotes the most general $\Lambda_\alpha x_{\phi\alpha}$ [$x_{\phi\alpha} = x_\alpha$ or x_α']; the generalized distributive law justifies this expansion. Each f_ϕ either is

O or covers O, for if $x_\alpha < f_\phi$, then clearly (since $x_{\phi\alpha} = x$ would imply $f_\phi \leqq x_\alpha$) $x_{\phi\alpha} = x'_\alpha$, and so $x_\alpha = x_\alpha \frown f_\phi \leqq x_\alpha \frown x'_\alpha = O$. Hence the $f_\phi > O$ are "points"; this is Tarski's central idea.

Moreover each $x_\alpha = x_\alpha \frown I = x_\alpha \frown Vf_\phi = V(x_\alpha \frown f_\phi)$ is the join of the $f_\phi > O$ which it contains (for either $f_\phi \leqq x_\alpha$ and $x_\alpha \frown f_\phi = f_\phi$, or $f_\phi \leqq x'_\alpha$ and $x_\alpha \frown f_\phi = O$). Again, a join $g_S = V_S f_\phi$ of points contains no point p not in S, since

$$g_S \frown p = \underset{S}{V}\,(f_\phi \frown p) = \underset{\phi \in S}{V}\, O = O,$$

by the first generalized distributive law. This establishes a one-one order-preserving correspondence, or isomorphism, between the sets S of "points" $f_\phi > O$, and the elements g_S of A, q.e.d.

115. Orthocomplemented lattices. In this section and the next, we shall consider converses of Theorem 6.1.

We already know (Corollary 2 of Theorem 5.1) that unicity of *relative* complements is equivalent to distributivity. Also (Theorem 4.5), in *modular* lattices, unicity of complementation implies distributivity. It is not known (and this is an outstanding unsolved problem of lattice theory) whether or not in general, existence and unicity of complementation imply distributivity. However, one can prove this in two special cases: (1) the case that complementation is orthocomplementation, and (2) the case that L is atomic.

LEMMA 1: *If every $a \in L$ has a unique complement a', and if $a \to a'$ is a dual automorphism, then $b \geqq a$ implies $(b \frown a') \smile a = b$ and dually $(a \smile b') \frown b = a$.*

Proof: Set $c = b \frown a'$; evidently $c \frown a = O$. Also $(c \smile a)' \frown b = [(b' \smile a) \frown a'] \frown b = (b' \smile a) \frown (a' \frown b)$, which is O since $b' \smile a$ and $a' \frown b$ are orthocomplements. Again, since $c \smile a = (b \frown a') \smile a \leqq b \smile b = b$, clearly $(c \smile a)' \smile b \geqq b' \smile b = I$. We conclude $(c \smile a)' = b'$, whence $c \smile a = b$, which is our first conclusion. The second follows by duality.*

Now suppose $y \frown x = a$, $y \smile x = b$, and form $c = b' \smile (x \frown a')$. Using Lemma 1 twice, and $y = a \smile y = b \frown y$ besides, we get

$$c \smile y = b' \smile (x \frown a') \smile a \smile y = b' \smile x \smile y = b' \smile b = I,$$

$$c \frown y = [b' \smile (x \frown a')] \frown b \frown y = x \frown a' \frown y = a' \frown a = O.$$

It follows that for fixed $a \leqq x \leqq b$, y is the (unique) complement of $b' \smile (x \frown a')$. Hence even *relative* complements are unique, and we conclude

THEOREM 6.11: *If every $a \in L$ has a unique complement a', and if $a \to a'$ is a dual automorphism, then L is a Boolean algebra.*

Again, let L be any lattice, such that every $a \in L$ has at least one complement a', and $a \frown x = O$ implies $x \leqq a'$ for all a'. Then any two complements of

* Stated another way our conclusion is that $b \frown a'$ is a relative complement of a in $L(O, b)$, and $a \smile b'$ is one of b in $L(a, I)$. Cf. Theorem 4.1.

the same a contain each other, and so complementation is unique. Also, $a \leqq b$ implies $a \frown b' = O$ and so $b' \leqq a'$. It is now a corollary of Theorem 6.11 that L is a Boolean algebra.*

116. Complete, atomic Boolean algebras. Again, let L be any complete, atomic† lattice. Thus L might be any finite lattice or any lattice of finite dimensions.

THEOREM 6.12: *If each element of L has only one complement, then L is isomorphic with the Boolean algebra of all subsets of its points.*

Proof: To each set S of points p of L, associate the join $x(S)$ of the p in S, and the meet $y(S)$ of the complements p' of $p \in S$. It will follow by generalized associativity (L*1, L*2), that $x(S \smile T) = x(S) \smile x(T)$ and $y(S \smile T) = y(S) \frown y(T)$. Again, the complement of $x(I)$ can contain no point;‡ hence it is O, and $x(I) = I$. Dually, $y(I) = O$.

Again, the complement p' of any point p is covered by I; that is, $x \geqq p'$ implies $x = p'$ or $x = I$. For if $x \geqq p$, then $x \geqq p \smile p' = I$; and if not, then $x \frown p < p$ will be O while $x \smile p \geqq p' \smile p = I$, whence $x = p'$. Hence (1) if p and q are distinct points, then $p \leqq q'$. For unless $p \leqq q'$, $p \frown q' = O$ and $p \smile q' = I$ (since p covers O and I covers q')—implying $q' = p'$ and so $q = p$. It is a corollary of (1) that (2) $x(S) \leqq y(S')$.

From the crucial inequality (2) we infer

(3) $x(S) \frown x(S') \leqq y(S') \frown y(S) = y(S \smile S') = y(I) = O$.

Thus $x(S)$ contains no point not in S, whence distinct sets S determine distinct $x(S)$, and the partially ordered system of the $x(S)$ is isomorphic with the algebra of all sets S.

It remains to show that every member a of L is an $x(S)$. But denote by S the set of points in a given $a \in L$. Evidently $a \frown x(S')$ will contain only points in S and in S'; hence $a \frown x(S') = O$. On the other hand, $a \smile x(S') \geqq x(S) \smile x(S') = x(S \smile S') = I$; hence a is the unique complement of $x(S')$. But this is $x(S)$, by (3) and the equality $x(S) \smile x(S') = I$, completing the proof.

Combining with Theorems 6.1 and 6.10, we get

COROLLARY 1: *A complete, atomic lattice L is a Boolean algebra if and only if each element of L has a unique complement.*

COROLLARY 2: *A complete Boolean algebra A is atomic if and only if it is completely distributive.*

117. Boolean rings. Stone has shown that one can subsume the theory of

* This conclusion is due to Huntington [1]. We note that the assumption that $a \frown x = O$ implies $x \leqq a'$ for *all* a' is not redundant; there exist lattices not Boolean algebras in which to each a corresponds an a' such that $a \frown a' = O$, $a \smile a' = I$, and $a \frown x = O$ implies $x \leqq a'$.

† Only the assumption that every element except O contains a point is used in the proof! The results of this section were obtained jointly with Morgan Ward (*A characterization of Boolean algebras*, Annals of Math., 40 (1939), 609–10).

‡ We are letting I denote both the biggest element in L and the set of all p.

Boolean algebras under the general theory of rings—actually, of commutative rings of characteristic two, in the usual sense (cf. van der Waerden, [1], p. 87).*

To see this, recall that any Boolean algebra is isomorphic with a field of sets, and recall also the notion of the "characteristic function" f_X of a set X in a space I: the function defined on the points $p \in I$, and satisfying $f_X(p) = 1$ or $f_X(p) = 0$ according as $p \in X$ or $p \in X'$. Then $f_X f_Y = f_{X \frown Y}$, and also $f_X + f_Y \equiv f_{(X \frown Y') \smile (X' \frown Y)} \bmod 2$.

Thus relative to *meets* $XY = X \frown Y$ and *symmetric differences* $X + Y = (X \frown Y') \smile (X' \frown Y)$, the elements of any Boolean algebra A form a ring $R(A)$ of characteristic two, with unit I satisfying $IX = XI = X$ for all X. Moreover in $R(A)$, $XX = X$ and $XY = YX$ identically: the ring is commutative, and all its elements are idempotent.

Actually, in any ring, the identity $xx = x$ implies

$$x + y = (x + y)(x + y) = xx + yx + xy + yy = x + y + yx + xy$$

and so it implies $xy + yx = 0$. Setting $x = y$, we get $x + x = 0$, and $x = -x$. Using this, we get $xy - yx = 0$, and so $xy = yx$. Hence $xx = x$ implies both $xy = yx$ and $x + x = 0$. This leads one to state

Definition 6.2: *A "Boolean ring" is a ring whose elements are all idempotent.*

Conversely, given a Boolean ring R with unit 1, if one defines $x \geqq y$ to mean $xy = y$, then (1) $1 \geqq x \geqq 0$ for all x, by definition of 1 resp. 0, (2) $x \geqq x$ by idempotence, (3) if $x \geqq y$ and $y \geqq x$, then $x = yx = xy = y$ (by commutativity), (4) if $x \geqq y$ and $y \geqq x$, then

$$x = xy = x(yz) = (xy)z = xz$$

and $x \geqq z$, (5) $x \geqq xy$, since $x(xy) = (xx)y = xy$, and (6) similarly (using commutativity) $y \geqq xy$, (7) if $x \geqq z$ and $y \geqq z$, then $xyz = xz = z$ and so $xy \geqq z$, (8) the correspondence $x \rightleftarrows 1 - x$ is a dual automorphism of period two; since $xy = y$ implies $(1 - y)(1 - x) = 1 - y - x + xy = (1 - x)$, it inverts inclusion, and it is obviously of period two.

Hence our definition makes R into a partially ordered set with O and I (by (1)–(4)), in which $x \frown y$ exists and is xy (by (5)–(7)), and (by (8)) $x \smile y$ exists and is $1 - (1 - x)(1 - y) = x + y - xy$. Also, if we set $x' = 1 - x$, then $x \frown x' = x(1 - x) = 0$ and $x \smile x' = x + (1 - x) - x(1 - x) = 1$; hence our

* Historical note: Boole originally characterized Boolean algebra as the algebra of 0 and 1 ([1], p. 37), but he did not appreciate the significance of addition modulo two. Stone was not the first person to note a relation between Boolean algebras and rings of characteristic two (cf. especially P. J. Daniell, *The modular difference of classes*, Bull. Am. Math. Soc., 23 (1916), 446–50; Gegalkin, Math. Sbornik, 35 (1928), 311–73; O. Frink, *On the existence of linear algebras in Boolean algebras*, Bull. Am. Math. Soc., 34 (1928), 329–33; H. Whitney, *Characteristic functions and the algebra of logic*, Annals of Math., 34 (1933), 405–14). But Stone was the first to realize the identity between the theory of Boolean algebras and that of a clear-cut family of rings (cf. Definition 6.2).

definition makes R into a complemented lattice. Finally,

$$x \frown (y \smile z) = x(y + z - yz) = xy + xz - xyz$$
$$= xy + xz - xyxz = (x \frown y) \smile (x \frown z),$$

and so R is a Boolean algebra. We conclude

THEOREM 6.13: *There is a one-one correspondence between Boolean algebras and "Boolean" rings (rings whose elements are idempotent) with unit. Under this, inclusion corresponds to divisibility, lattice meets to ring products, symmetric differences to ring sums, O to* 0, *and I to* 1.

Again, $a \geqq b$ corresponds to $a|b$, $a \frown b = ab = \text{l.c.m.}\ (a, b)$, and $a \smile b = a + b - ab = \text{h.c.f.}\ (a, b)$.

The same correspondence makes subalgebras correspond to subrings, ideals to ideals, and prime ideals to prime ideals. It is also a part of a wider correspondence between *general* Boolean rings and "generalized" Boolean algebras. (By this is meant a distributive lattice with O but without I, in which relative complements exist.) For further details, cf. Stone [3], Theorem 4 ff.

118. Applications to topology: Boolean spaces. We recall from §104 that a bicompact T_1-space is characterized topologically by any "basis" of closed sets. But a closed set has a closed complement if and only if it disconnects the space; hence a space has a Boolean algebra for a basis of closed sets, if and only if it is *totally disconnected.**

Moreover any Boolean algebra has Wallman's disjunction property. Hence *every* Boolean algebra can be regarded as a basis of closed sets of a totally disconnected bicompact T_1-space (or "Boolean space")—and conversely, the open and closed subsets of any Boolean space I form a Boolean algebra $A(I)$ which characterizes I.

We note that since under this correspondence, the points of I appear as the different prime ideals of $A(I)$, I affords the "perfect" representation of $A(I)$ by a field of sets.

Stone has also proved ([4], p. 393) the curious fact that if I is the Cantor discontinuum, then $A(I)$ is the "free" Boolean algebra with countably many generators.

119. Elementary figures. The rest of this chapter will be concerned with special Boolean algebras which play an important role in set theory. Cf. also C. Carathéodory, *Vorlesungen über reelle Funktionen*, Leipzig, 1927, Chap. V, and *Entwurf für eine Algebraisierung des Integralbegriffs*, S.-B. Bayer. Akad. (1938), 27–69.

* The usual definition of total disconnectedness, that any two points fall in complementary closed sets, is equivalent in "regular" spaces to the assertion that any closed set is the intersection of open closed sets. The results of the present section are due to Stone, who has shown that generalized Boolean algebras correspond to totally disconnected locally bicompact spaces.

First, let I_n denote the unit n-cube—i.e., the set of all points $(x_1, \cdots, x_n)$ $[0 \leqq x_k \leqq 1]$—and let A_n denote the Boolean algebra of all subsets of I_n. By a "measure function," is meant a non-negative, modular functional, defined on a subalgebra of A_n, satisfying $m(O) = 0$, $m(I) = 1$. This convenient abstraction includes the familiar concepts of length, area and volume as special cases.

The simplest figures in I_n are the closed "generalized rectangles" of points $(x_1, \cdots, x_n)$ satisfying $a_k \leqq x_k \leqq b_k$ $[k = 1, \cdots, n]$. If measure is invariant under translation, the measure of such a figure must equal the product $\prod_{k=1}^{n} (b_k - a_k)$ of the lengths of its sides. The subalgebra of A_n which is generated by these generalized rectangles consists of the so-called "elementary figures." Any elementary figure can be decomposed into generalized rectangles which overlap only on sub-rectangles of measure zero—and so its measure must be (under our hypotheses) the sum of the measures of these.

Thus one can construct a quasi-metric Boolean subalgebra E_n of A_n, consisting of the elementary figures. If one ignores elementary figures of less than n dimensions—or, what is the same thing, elementary figures of measure zero one obtains by Theorems 3.10 and 6.6, a *metric* Boolean algebra E_n/Z_n of considerable interest.†

120. Jordan sets. We can now enlarge our class of measurable sets, by using a very general construction.

Indeed, let S be any quasi-metric subalgebra of any Boolean algebra A, and let the functional relative to which S is quasi-metric be $m[x]$. Then one can define two new functionals on A as follows,

$$\underline{m}[a] \equiv \sup_{x \leqq a} m[x], \qquad \bar{m}[a] \equiv \inf_{x \geqq a} m[x].$$

Clearly $\underline{m}[a] \leqq \bar{m}[a]$ for all $a \in A$—and $\underline{m}[a] = m[a] = \bar{m}[a]$ for all $a \in S$.

THEOREM 6.14: *The elements a for which* $\underline{m}[a] = \bar{m}[a]$ *form a subalgebra M of A, quasi-metrized by* $\underline{m} = \bar{m}$.

Proof: Clearly $a \in M$ if and only if, given $\delta > 0$, one can find $x, x^* \in S$ satisfying $x \leqq a \leqq x^*$ and

$$| x^* - x | \equiv m[x^* \smile x] - m[x^* \frown x] = m[x^*] - m[x] < \delta.$$

But if this is true, and the same conditions hold for $y \leqq b \leqq y^*$ $[y, y^* \in S]$, then $| x' - x^{*\prime} | < \delta$, and by Theorem 3.10,

$$| x^* \frown y^* - x \frown y | < 2\delta, \qquad | x^* \smile y^* - x \smile y | < 2\delta,$$

where $x^{*\prime} \leqq a' \leqq x'$, $x \frown y \leqq a \frown b \leqq x^* \frown y^*$, and $x \smile y \leqq a \smile b \leqq x^* \smile y^*$. It follows that $a' \in M$, $a \frown b \in M$, and $a \smile b \in M$, q.e.d.

Remark: It is easy to show that if $m[O] = 0$ and $m[I] = 1$, then $\underline{m}[a] = 1 - \bar{m}[a']$—and as a corollary of this that $a \in M$ if and only if $\underline{m}[a] + \underline{m}[a'] = 1$.

† It is easy to show that if any non-negative, modular functional is defined on any Boolean algebra, then the elements with $m(X) = m(O)$ form an "ideal."

In the case discussed in the preceding section, setting $A = A_n$ and $S = E_n$, we get Jordan's theory of measure: the sets in M are the "measurable" sets, and $\underline{m} = \bar{m}$ gives their "measure." If X is *any* set, measurable or not, $\underline{m}[X]$ is called its *inner* (Jordan) measure, and $\bar{m}[X]$ its *outer* Jordan measure.

Once more, by ignoring sets of measure zero, we get an interesting metric quotient-algebra. This is the same for all dimensions (although we shall not prove it)—which suggests that it might be interesting to characterize it abstractly. This will be done in §122.

121. Borel sets. From almost every point of view, Jordan's theory of measure is improved by the twentieth century theory of Borel and Lebesgue, which has been perfected by Carathéodory.

To obtain this, we define a "Borel set" as a member of the σ-subalgebra of A_n generated by closed generalized rectangles. The analogy between this and our definition of an "elementary figure" is clear. It is easy to show that all open generalized rectangles (being differences between closed generalized rectangles) are elementary figures and so Borel sets. It follows that all open sets, and hence all closed sets, are Borel sets. Thus we might equally well have defined a Borel set as any member of the σ-subalgebra of A_n generated by open and closed sets, and had a concept which extended to the most general topological space.

It is a theorem of measure theory—whose proof will not be reproduced here*—that there is one and only one way to extend the measure functional from generalized rectangles to Borel sets, so that it remains additive and continuous. In this way we obtain a quasi-metric Boolean σ-algebra B_n. By ignoring Borel sets of measure zero, we obtain a metric Boolean σ-algebra B_n/Z_n.

This will be characterized abstractly in §122, from which it will appear that it is independent of n.

122. Lebesgue measurable sets. If now we consider B_n as a subalgebra of A_n, and apply Theorem 6.14, we get the class M_n of Lebesgue measurable sets. This is the ultimate class of the modern theory of measure and integration.

Actually, one gets the same family of sets, if one replaces B_n by the smaller class of unions of countable open (or closed!) rectangles: it is a well-known theorem of measure theory that one gets in this way a σ-subalgebra of A_n—which must thus include B_n, and so M_n.

If one ignores so-called "null sets" (sets of measure zero), one gets a metric quotient-algebra M_n/N_n, which we shall now study in detail.

Firstly, M_n/N_n is metrically isomorphic to B_n/Z_n. For every measurable set is equivalent modulo a null set to an intersection of countable open sets,† which is a Borel set of class G_δ.

* Indeed, it is much easier to prove the theorem for the more general category of "measurable" sets defined below—and then to show that this category is a Boolean σ-algebra including generalized rectangles as elements.

† Namely, let a sequence of open sets be chosen, whose measures exceed that of the given set by smaller and smaller amounts and yet which contain it. Their intersection will satisfy our conditions.

THEOREM 6.15: *The algebra M/N of measurable subsets of the unit cube of Cartesian n-space, modulo sets of measure zero, is obtained by completing metrically the limit as $k \to \infty$ of the Boolean algebras B^{2^k}, metrized by the formula** $m[x] = d[x]/d[I]$.

Remark: It is a corollary that it is independent of n.

Proof: Firstly, M/N is complete. For let $\{u_n\}$ be any sequence of elements of M, such that $\delta(u_m, u_n) \to 0$ as $m, n \to \infty$. Then $\{u_n\}$ contains a subsequence $\{v_k\} = \{u_{n(k)}\}$ with $\delta(v_k, v_{k+1}) \leqq 2^{-k-1}$. Form the "symmetric differences" $s_k \equiv (v_k \smile v_{k+1}) \frown (v_k \frown v_{k+1})'$. Then since M is a σ-field of sets, $w_n \equiv v_n \smile V_{k=0}^{\infty} s_{n+k}$ is in M; also $w_n \geqq w_{n-1}$. The intersection v of the w_n will be in M, and since measure is continuous $\delta(v_k, v) = \lim_{n\to\infty} \delta(v_k, w_n)$. But if $n \geqq k$, then

$$\delta(v_k, w_n) \leqq \delta(v_k, v_n) + \delta(v_n, w_n) \leqq 2^{-k} + 2^{-n}.$$

Hence $\delta(v_k, v) \leqq 2^{-k}$ and $\{u_n\}$ converges metrically to v, proving M complete.

Now let π_k denote the decomposition of I into cubes of side 2^{-k}; the elementary figures composed of these will constitute a subalgebra of M/N isomorphic with $B^{2^{kn}}$, in which $m[x] = d[x]/d[I]$. The union of the subalgebras so defined is evidently $\lim_{k\to\infty} B^{2^k}$; moreover any elementary figure can be approximated arbitrarily closely by members of this union—and hence so can any measurable set. That is, M/N is obtained by completing this union metrically, which was what we wished to prove.

A variation of the proof of Theorem 6.15 shows that the metric Boolean algebra of measurable subsets of any region of unit measure in Cartesian n-space, modulo null sets, is metrically isomorphic with M/N. Hence the same is true for any region of finite positive measure, if the measure function is multiplied by a suitable constant.

Hence if $O < x < I$ in M/N, the $t \leqq x$ form a Boolean algebra isomorphic to M/N—but with all distances shrunk in a fixed ratio. Moreover M/N is the product of the sublattice of $t \leqq x$ and the sublattice of $t \leqq x'$. Since this decomposition of M/N into factors isomorphic with x is independent of x, we conclude

THEOREM 6.16:† *The group of automorphisms on M/N is transitive on the elements not O or I.*

Returning to Theorem 6.15, one can characterize abstractly the metric quo-

* By B^{2^k} is meant the finite Boolean algebra of dimensions 2^k and order 2^{2^k}. The analogy between this and von Neumann's construction of a "continuous geometry" is plain.

† A similar argument shows that the group of isometric automorphisms of M/N is transitive on elements of the same measure. Thus M/N is highly homogeneous.

Actually (V. Glivenko) any isometry leaving O fixed is an automorphism: $x \smile y$ is the element t furthest from O among those satisfying $\rho(x, t) + \rho(t, y) = \rho(x, y)$. The other isometries are the products of isometric automorphisms with transformations $x \to x + a$ in the sense of Boolean rings.

tient-algebra of Jordan sets modulo null sets, as follows. Metrize $S \equiv \lim_{k\to\infty} B^{2^k}$ by making $m(x)$ be $d[x]/d[I]$. Then adjoin those elements a of M/N which can be approached both by $x \geqq a$ and by $y \leqq a$ in S.

123. Related systems. Various other metric Boolean algebras of importance in set-theory are closely related to M/N.

Thus consider the algebra M^*/N of all Borel sets of finite measure in Cartesian n-space, modulo sets of measure zero. Relative to measure, this is a metrically complete generalized Boolean algebra; moreover it is easy to characterize M^*/N abstractly.

To do this, regard space as the sum of countable regions of measure one, and every set of finite measure as the limit of its parts on successively larger sums of these regions. Following this idea, one sees that M^*/N contains a sequence of sublattices M/N, $(M/N)^2$, $(M/N)^3$, $\cdots$, each containing the preceding in the sense of metric isomorphism. By completing metrically the union (envelope) of this expanding sequence, one obtains M^*/N. Moreover the same idea applies to any region of infinite measure in Cartesian n-space.

Again, consider Daniell's theory† of measure on the "infinite-dimensional torus" of points $x = (x_1, x_2, x_3, \cdots)$ $[x_i \bmod 1]$. Let π_n denote the slicing up of the torus into the $2^{n(n-1)/2}$ "generalized rectangles"

$$R\colon k(R)/2^{n-i} \leqq x_i \leqq (k(R)+1)/2^{n-i} \qquad [i = 1, \cdots, n].$$

Beginning with these rectangles, one can get as in the case of Lebesgue measure an algebra of "measurable subsets modulo sets of measure zero," metrically isomorphic with M/N.

To take a third case, in the so-called Radon-Stieltjes theory of integration, one begins with a modification of M/N in which lumps or "atoms" of positive measure are allowed.

The two remaining systems arising in measure theory can only be characterized abstractly in terms of the rather complicated Moore-Smith concept of limit. These systems are respectively: Hausdorff's sets of (finite) p-dimensional measure in Cartesian q-space, and an apparently unformulated theory of measure for tori having uncountably many coordinates. The first system is analogous to M^*/N, but the simple sequence of $(M/N)^k$ is replaced by a "directed set" of partly overlapping $(M/N)^k$; the second system is analogous to M/N, but the simple sequence of B^{2^k} is replaced by a directed set of uncountable powers of B.

From each of the above theories of measure, one gets a theory of integration by replacing single-valued point-functions by multiple-valued set-functions: given an ordinary function $f(p)$, one defines $F(S)$ as assuming for each set S all the values assumed by $f(p)$ for any $p \in S$.

124. Open sets modulo nowhere dense sets. Let I be any topological space,

† P. J. Daniell, *Integrals in an infinite number of dimensions*, Annals of Math., 20 (1919), 281–8; B. Jessen, Seventh Scand. Math. Congr. Oslo (1929), 127–38; R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, American Mathematical Society Colloquium Publications, vol. 19, 1934, Chap. IX.

and A the algebra of all subsets of I. We have already mentioned two complete, distributive sublattices of A: the family of closed sets, and the family F of open sets (complements of closed sets). Clearly these are dually isomorphic.

Now define a "pseudo-complement"† of an element a of a lattice L, as an element a^* such that $a \frown x = O$ if and only if $x \leqq a^*$. If x is an open set, then $a \frown x = O$ if and only if $\bar{a} \frown x = O$, and so every $a \in F$ has a pseudo-complement in F, namely, the set-complement $(\bar{a})' = a^*$ of the closure $\bar{a}$ of a. We can infer

Lemma 1: *The open sets of I form a pseudo-complemented lattice F. Two members of F are* mutually *pseudo-complementary when each is the set-complement of the closure of the other.*‡

It is clear that a and a^* are mutually pseudo-complementary, if and only if $(a^*)^* = a$ —and that otherwise (since $a \frown a^* = O$) $(a^*)^* > a$. But $(a^*)^* = (((\bar{a})')^-)'$ is by definition of interior, the *interior* of the closure of a. Hence an open set is a member of a mutually pseudo-complement pair, if and only if it is the interior of its closure—i.e., a so-called "regular" open set (Stone [4]).

Now recall the notion of a nowhere dense set, as a set x such that if b is open and non-void, then $b \frown x'$ contains an open non-void subset. It is easy to show that nowhere dense sets form an *ideal* D in A —hence we can define a quotient-lattice F/D of open sets modulo nowhere dense sets. Incidentally, no open set is nowhere dense.

Lemma 2: *The difference between any open set a and $(a^*)^*$ is nowhere dense.*

Proof: If b is open and non-void, then

$$b \frown [(a^*)^* \frown a']' = b \frown [((\bar{a})')^- \smile a] \geqq (b \frown a^*) \smile (b \frown a)$$

which is open, and cannot be void since $b \frown a = O$ implies $b \frown a^* = b > O$. Hence $(a^*)^* - a$ is nowhere dense.

Lemma 3: *If the difference between $(a^*)^*$ and $(b^*)^*$ is nowhere dense, then $(a^*)^* = (b^*)^*$.*

Proof: The part of $(a^*)^*$ not in the closure of $(b^*)^*$ is an open set contained in the difference between $(a^*)^*$ and $(b^*)^*$; being nowhere dense, it is void. Being void, the closure c of $(b^*)^*$ contains $(a^*)^*$ —and hence so does the interior $(b^*)^*$ of c. Similarly, $(a^*)^* \geqq (b^*)^*$, whence by P2 the two are equal, q.e.d.

125. Borel sets modulo sets of first category. Again, recall the definition of a set of first (Baire) category, as a sum of countable nowhere dense sets; it is

† G. Birkhoff [1], p. 459—where it is pointed out that any finite distributive lattice is pseudo-complemented, and that no element can have more than one pseudo-complement. The notion of a pseudo-complement also appears in Stone's calculus of ideals. Cf. also §161.

‡ An equivalent condition is that they be disjoint and their sum be I minus their common boundary.

obvious that the sets of first category in any topological space I are a σ-ideal, J_c. Moreover it is well-known* that if I is any region of ordinary space, then (1) no open subset of I is of first category, and (2) any Borel set not of first category is congruent to an open set modulo a set of first category.

It follows by Lemma 2 that every Borel set is congruent to a regular open set. But using (2) in the proof of Lemma 3, we see that no two distinct regular open sets are congruent modulo sets of first category. Hence

THEOREM 6.17: *The following systems are isomorphic: the system of regular open subsets of I, the lattice of open sets modulo nowhere dense sets, and the Boolean σ-algebra B/J_c of Borel sets modulo sets of first category.*

THEOREM 6.18: *The completion by cuts of* $\lim_{k\to\infty} B^{2^k}$ *is isomorphic with the systems described in Theorem* 6.17.†

Proof: A dyadic decomposition of I modulo nowhere dense sets shows that it contains $\lim_{k\to\infty} B^{2^k}$. And any pair of pseudo-complementary open sets defines a cut in this—and conversely. We omit the details.

Since the Boolean algebra just described is the same for all open sets in space, we can prove, as in Theorem 6.16 that the group of automorphisms of B/J_c is transitive on the elements not O or I.

In spite of the resemblance between them, B/J_c and M/N are non-isomorphic. For B/J_c contains a countable family $\{x_k\}$ (namely, intervals bounded by points with rational coordinates), such that no set of y_k with $O < y_k \leqq x_k$ for all k has an upper bound less than I—the complement of such an upper bound will be nowhere dense. Whereas M/N contains no such family of x_k—one can always make the total measure of the y_k arbitrarily small.

* C. Kuratowski, *Topologie*, p. 51, line 1. The analogy between measure and category has often been remarked; we shall now look at the analogy from the point of view of lattice theory.

† The completeness of B/J_c, and its non-isomorphism with M/N, were first proved by S. Ulam and the author cf. Proceedings Oslo Congress (1936), vol. 2, p. 37. Another proof is given by J. von Neumann [2], Part IV. Its abstract characterization was due to H. M. MacNeille.

CHAPTER VII

APPLICATIONS TO FUNCTION THEORY

126. Introduction. Function spaces, unknown thirty years ago, have shed so much light on the theory of functions that their study is now a recognized branch of mathematics. The present chapter is concerned with the relation of partial ordering in function spaces.

This topic has been insufficiently exploited; it is not even mentioned in the two most widely read books on function spaces.* In fact, the relation $f \leqq g$ cannot even be defined in terms of the concepts used in these books.† Neither can the following subsidiary notions: positive function, positive part of a function, supremum, infimum, lim sup, lim inf, oscillation, majorant—although they play an important role in the theory of real functions.

In Chapter IX, we shall show that these notions play an essential role in the theory of dependent probabilities (stochastic processes); in the present chapter, we shall consider results applying purely to the theory of functions.

127. Main results. In the first place, we can show that every known function space forms a lattice with respect to a natural partial ordering—and that most of them form (conditionally) complete lattices. If the function space is a vector space, the resulting lattice is distributive.

Again, we can not only define "relative uniform convergence" in the classic sense of E. H. Moore, but we can show that it gives the usual metric topology ("strong topology"), for all known Banach spaces. In this sense we can define distance (to within a bounded factor) in terms of order—although the converse is impossible.‡ We can also show that "convergence in measure" is star-convergence in the lattice-theoretic sense.

Further, we can show that an additive functional carries bounded sets into bounded sets, if and only if it is the difference of two positive functionals. These "bounded" additive functionals form what may be called the "conjugate space" of a given partially ordered function space. This terminology is especially suitable, since it contains as special cases (1) Banach's notion of a conjugate space, and (2) the notion of a "Dualraum" introduced by Koethe and Toeplitz.

Finally, we can characterize abstractly the decompositions of partially ordered function spaces—in particular, the components form a Boolean algebra. A

* Namely, M. H. Stone's *Linear Transformations in Hilbert Space and their Applications to Analysis*, New York, 1932, and S. Banach's *Théorie des Opérations Linéaires*, Warsaw, 1933.

† Proof: Hilbert space, and the other better known Banach spaces, admit automorphisms (isometric linear transformations) which mix positive functions with non-positive functions.

‡ This indicates that order is more fundamental than Banach's "norm," and could be used as a basis for Banach's theory.

related result is that under suitable restrictions, every bounded additive functional decomposes its domain into components N^-, N^0, and N^+, on which it is negative, zero, and positive respectively.*

128. Partially ordered function spaces. The cornerstone of our theory is the following definition,†

DEFINITION 7.1: *A "partially ordered linear space" is a (real‡) linear space, some of whose elements f are "non-negative" (written $f \geqq 0$), and in which*

F1: *If $f \geqq 0$ and $\lambda \geqq 0$, then $\lambda f \geqq 0$.*
F2: *If $f \geqq 0$ and $-f \geqq 0$, then $f = 0$.*
F3: *If $f \geqq 0$ and $g \geqq 0$, then $f + g \geqq 0$.*

THEOREM 7.1: *If $f \geqq g$ is defined to mean $f - g \geqq 0$, then any partially ordered linear space is a partially ordered system. The transformations $f \to f + a$ and $f \to \lambda f$ $[\lambda > 0]$ induce automorphisms on this system; the transformation $f \to -f$ is an "involution," or dual automorphism of period two.*

Proof: For any non-negative g, $0 = 0g \geqq 0$ by F1. Hence $0 \geqq 0$, and $f - f \geqq 0$ for all f, proving P1. Again, if $f - g \geqq 0$ and $g - f = -(f - g) \geqq 0$, then $f - g = 0$ by F2 and $f = g$, proving P2. Finally, if $f - g \geqq 0$ and $g - h \geqq 0$, then by F3, $f - h = (f - g) + (g - h) \geqq 0$, proving P3.

The transformations described preserve or invert order as asserted, since $(f + a) - (g + a) = f - g$, $\lambda f - \lambda g = \lambda(f - g)$, and $-f - (-g) = g - f$. But they are one-one, by the general theory of linear spaces; similarly, $f \to -f$ is of period two.

Remark: The conditions of Theorem 7.1 give a definition of partially ordered linear spaces, equivalent to Definition 7.1.

129. Vector lattices. Most important function spaces are "vector lattices," in the sense of

DEFINITION 7.2: *A partially ordered linear space will be called a vector lattice, if and only if the partial ordering of Theorem 7.1 defines a lattice. For any f, $f \smile 0 = f^+$ will be called the "positive part," and $f \frown 0 = f^-$ the "negative part" of f.*

THEOREM 7.2: *For a partially ordered linear space to be a vector lattice, it is sufficient that every element have a positive part—that $f \smile 0$ exist for all f.*

Proof: By Theorem 7.1, $(f - g)^+ + g$ will be $f \smile g$, and $-(-f \smile -g)$ will be $f \frown g$—and both will exist.

130. Complete vector lattices. When we speak of a σ-complete vector lat-

* These results were stated in a lecture given at Notre Dame University in February, 1938.

† Definitions 7.1–7.2 are essentially due to Kantorovitch [1] (and earlier announcements). Freudenthal [1] developed the same ideas independently, attributing his inspiration to F. Riesz [1].

‡ We are concerned with functions of a real variable alone, and with linear spaces (vector spaces) having real scalars alone. As is well-known, any linear space with complex scalars can also be represented as one with real scalars.

tice, we shall mean a vector lattice which is *conditionally* σ-complete resp. complete, regarded as a lattice—a vector lattice in which every *bounded* countable subset (resp. every bounded subset) has a supremum and an infimum. We do this because of the following

THEOREM 7.3: *Let L be any σ-complete vector lattice. Then*† (1) *if* $f \neq 0$, *the set* $\{nf\}$ *is unbounded, while* (2) *if* $f > 0$ *and* $\lambda_n \downarrow 0$, *then* $\lambda_n f \downarrow 0$.

Proof of (1): If $a \leqq nf \leqq b$ for all n, then by Theorem 7.1 $nf^+ \leqq b^+$ for all n, and $\{nf^+\}$ is bounded. But clearly $\sup \{nf^+\} = \sup \{2nf^+\} = 2 \sup \{nf^+\}$; therefore $\sup \{nf^+\} = 0$, and $f^+ \leqq 0$. Dually, $f^- \geqq 0$, whence $0 \leqq f^- \leqq f \leqq f^+ \leqq 0$, and $f = 0$.

Proof of (2): As above, $\inf \{\lambda_n f\} = \inf \{2\lambda_n f\} = 2 \inf \{\lambda_n f\}$, whence as above, $\inf \{\lambda_n f\} = 0$.

It is a corollary that no partially ordered linear space forms a σ-complete lattice: if it is conditionally σ-complete, it contains unbounded countable sets. The most that we can hope for is that *bounded* (countable) sets should have suprema and infima.

Remark: The conclusions of Theorem 7.3 do not hold in all vector lattices; the hypothesis of σ-completeness is not redundant. For example, they fail in the Cartesian plane if $(x, y) \leqq (x^*, y^*)$ is defined to mean: either $x < x^*$, or $x = x^*$ and $y \leqq y^*$. Thus they do not hold in Cartesian n-space "lexicographically" ordered.

131. Simplest examples. The simplest example of a vector lattice, and the prototype for all vector lattices, is of course the real number system R. This is a one-dimensional linear space, having the positive numbers and zero as its "non-negative" elements.

THEOREM 7.4: *Any direct union of vector lattices is a vector lattice.*

The proof can be left to the reader; it involves only a component-by-component verification of F1–F3. It is equally easy to show that if the original lattices are (conditionally) complete, then so is their union.

It is corollary that the spaces R^n of all vectors $[x_1, \cdots, x_n]$ with n real components, R^ω of all infinite sequences $x_1, x_2, x_3, \cdots$ of real numbers, and R^c of all real functions defined on the interval [0, 1] are complete vector lattices.

The notation for these spaces is self-explanatory, if one admits the power notation of §16, and imagines ω and c as denoting countable infinity resp. the power of the continuum.

132. Other examples. It is clear that any subspace of a vector lattice which is a sublattice at the same time is a vector lattice itself. That is, if a subset contains with f and g also $f + g$, $f \smile g$, $f \frown g$, and every λf, then it is a vector lattice relative to the same operations.

These hypotheses are satisfied by the following sets of functions: the subset

† The first of these results is stated by Kantorovitch, Doklady, IV (1935), pp. 13–16, as the "Principle of Archimedes."

(B) of R^c consisting of all bounded functions; the subset (b) of R^ω consisting of all bounded sequences; the subset C of R^c consisting of all continuous functions; the subset (c) of R^ω consisting of all convergent sequences; by analogy, the set of continuous functions on any topological space; similarly, the set D of functions on any topological space which admit only a finite number of discontinuities. Again, so do: the set (M) of measurable functions (in the sense of Lebesgue) on the line; the subset (M^p) having summable pth powers; the set (l^p) in R^ω consisting of all sequences with finite $\sum_{k=1}^{\infty} | x_k |^p$.

So also does the set of functions Lip α. Finally, let Γ be any (infinite) group of one-one transformations of a space Σ into itself, and call a function "almost periodic" (in the sense of Bochner) when every infinite sequence of its transforms under Γ contains a uniformly convergent subsequence. Then the real functions on Σ almost periodic under Γ satisfy our hypotheses.*

Most of these examples can be found in Banach [1], whose notation we have adopted wherever convenient. The truth of our assertions concerning these sets of functions is in most cases well-known; in the others it can easily be proved using well-known function-theoretic arguments, and Theorem 7.2.

133. Semi-vector lattices. Several interesting classes of functions form lattices without forming linear spaces. They form what may be called "semi-linear spaces": vector addition, and multiplication by non-negative scalars can be performed, subject to the usual laws of the vector calculus.

For example, consider the sets (USC) and (LSC) of upper resp. lower semi-continuous functions defined on any region of the plane (or of any topological space!). These form a sublattice of R^c, and also a semi-linear subspace.

Again, consider the set (Con) of convex functions (or of continuous convex functions) defined on the real axis. It is a semi-linear subspace of R^c, and contains with any f and g, also $f \cup g$. It is even a lattice: since the supremum of any bounded family of convex functions is convex, we can apply Theorem 2.2 to define the "convex infimum" of functions f and g, as the supremum of the convex functions less than or equal to both f and g.

Precisely similar remarks apply to the set (SH) of subharmonic functions defined on any plane region, and we can define the "subharmonic infimum" of given functions likewise.

We shall see later that this apparently trivial absence of multiplication by negative scalars has purely lattice-theoretic consequences. Notably, it makes distributivity no longer necessary, and also continuity of the operations in the order-topology!

134. The Jordan decomposition. Any element of any vector lattice has a "Jordan decomposition" into its positive and negative parts.† That is, formally

* If $h = f \cup g$, and $h(aT_n)$ is any sequence of transforms of h, then we can choose a subsequence and a subsubsequence on which both f and g converge uniformly, whence so does h. Hence they form a sublattice of R^c. They do not, however, form a σ-sublattice.

† The phrase is due to Saks, *Théorie de l'Intégrale*, Warsaw, 1934, p. 9; cf. Jordan, *Cours d'Analyse*, vol. 1, p. 54. Cf. also F. Riesz, *Sur la décomposition des opérations fonctionelles*, Atti Congresso Bologna (1928), vol. 3, 143–8.

THEOREM 7.5: *We have* $f = f^+ + f^-$ *identically.*

Proof: By Theorem 7.1, $-(f \frown 0) = 0 \smile (-f) = (f \smile 0) - f$, that is, $-f^- = f^+ - f$. Now transpose f and f^-.

Hence $(g - h) + 0 = (g - h) \smile 0 + (g - h) \frown 0$. But under the lattice-automorphism (cf. Theorem 7.1) $x \to x + h$, this becomes

COROLLARY: *For all* g *and* h, $g + h = (g \smile h) + (g \frown h)$.

THEOREM 7.6:* *Any vector lattice is distributive.*

Proof: Relative complementation is unique, since if $a \smile x = a \smile y = u$ and $a \frown x = a \frown y = v$, then $a + x = u + v = a + y$ and $x = y$. Theorem 5.2 completes the proof.

COROLLARY 1: *The correspondences* $x \to x^+$ *and* $x \to x^-$ *define lattice-endomorphisms.*

Again, $0 = f + (-f) = f \smile -f + f \frown -f \geqq 2(f \frown -f)$, whence $f \frown -f \leqq 0$. Hence by Theorem 7.6, $0 = 0 \smile (f \frown -f) = (0 \smile f) \frown (0 \smile -f) = f^+ \frown (-f)^+$, and

COROLLARY 2: *The positive parts of* f *and* $-f$ *are always disjoint—in symbols,* $f^+ \frown (-f)^+ = 0$.

135. Absolute of an element. We shall now define the "absolute" $|f|$ of an element of a vector lattice, as $f \smile -f$. It behaves like an absolute value in many ways. Thus†

THEOREM 7.7: (1) $|f| > 0$ *unless* $f = 0$, (2) $|\lambda f| = |\lambda| \cdot |f|$, (3) $|f + g| \leqq |f| + |g|$, (4) $|f| = f^+ - f^-$, (5) $|f - g| = (f \smile g) - (f \frown g)$.

Proof: If $f \neq 0$, then $f \neq -f$ and

$$-[f \smile (-f)] = (-f) \frown f < f \smile -f.$$

Hence $0 < 2(f \smile -f) = 2|f|$, and $|f| > 0$, proving formula (1). Again, $|f| = |-f|$ by definition and L2, while for positive λ, $|\lambda f| = \lambda |f|$ by Theorem 7.1, proving (2). Thirdly, $|f| + |g|$ is an upper bound to both $f + g$ and $-f - g$, since its addends are to the addends involved. Hence

(3) $|f| + |g| \geqq (f + g) \smile (-f - g) = |f + g|$.

Again, using (1), $|f| = |f| \smile 0 = f \smile 0 \smile (-f) \smile 0 = f^+ \smile (-f)^+$. But $f^+ \smile (-f)^+ = f^+ + (-f)^+ - [f^+ \frown (-f)^+]$ by Theorem 7.5, Corollary. By Corollary 2 of Theorem 7.6, this is $f^+ + (-f)^+$, which is $f^+ - f^-$ by Theorem 7.1. This proves (4); now setting $f = g - h$, we get $|g - h| = (g - h)^+ - (g - h)^-$

* This result is due to Freudenthal [1], p. 642. It is not true in the semi-vector lattices of convex or of subharmonic functions.

† The results of the present section are due to Kantorovitch [1], who showed the power of the notion.

$= (g - h) \smile 0 - (g - h) \frown 0$. But by Theorem 7.1, $(g - h) \smile 0 = (g \smile h) + h$, and $(g - h) \frown 0 = (g \frown h) + h$; substituting and cancelling, we get (5).

THEOREM 7.8: *We have identically,*

$$|(f \smile g) - (f^* \smile g)| + |(f \frown g) - (f^* \frown g)| = |f - f^*|.$$

Proof (cf. Theorem 3.10 for analogy): The left-hand side is, by formula (5) of Theorem 7.7,

$$f \smile g \smile f^* - (f \smile g) \frown (f^* \smile g) + (f \frown g) \smile (f^* \frown g) - f \frown g \frown f^*.$$

But by the distributive law (Theorem 7.6), this is

$$(f \smile f^*) \smile g - (f \frown f^*) \smile g + (f \smile f^*) \frown g - (f \frown f^*) \frown g.$$

Hence, transposing and using the Corollary to Theorem 7.5, it is

$$(f \smile f^*) + g - (f \frown f^*) - g = (f \smile f^*) - (f \frown f^*) = |f - f^*|,$$

which completes the proof.

136. Normal subspaces and decompositions. It is known (cf. the Foreword) that any congruence relation on a linear space Σ is determined by the set S of elements congruent to 0. Moreover the different "quotient-spaces" Σ/S so determined have for "congruence modules" the different *subspaces* of Σ.

THEOREM 7.9:† *A subspace determines a* lattice-*homomorphism if and only if it contains with f all x satisfying $|x| \leqq |f|$.*

That is, this condition is necessary and sufficient for joins (and dually, meets) of elements congruent mod S to be congruent mod S. For if this is so, and $f \equiv 0\ (S)$, then $|f| = f \smile -f \equiv 0 \smile -0 = 0\ (S)$. Hence if $|g| \leqq |f|$ and $f \in S$, then $0 \leqq g^+ \leqq |f|$ so that $g^+ \in S$; dually, $g^- \in S$, and consequently $g = g^+ + g^- \equiv 0\ (S)$.

Conversely, if S contains with any f all x such that $|x| \leqq |f|$, then $|(f + h) \smile g - (f \smile g)| \leqq |h|$ by Theorem 7.8; hence substitution of $f + h$ $[h \in S]$ for f in $f \smile g$ changes $f \smile g$ by an element whose absolute is bounded by $|h|$, and hence which is in S. We conclude that the correspondence $\Sigma \to \Sigma/S$ is join-homomorphic; dually, it is meet-homomorphic, completing the proof.

DEFINITION 7.3: *A subspace of a vector lattice will be called "normal," when it satisfies the condition of Theorem* 7.9.

THEOREM 7.10: *Any normal subspace N is convex.*

Proof: If $a \in N$, $b \in N$, and $a \leqq x \leqq b$, then $|x| \leqq |x - a| + |a| \leqq |a| + |b| \in N$, and so $x \in N$.

† This result and Theorems 7.11–7.12 are due to the author [8] (*Dependent probabilities and spaces (L)*, Proc. Nat. Acad. Sci., 24 (1938), 155–9). The condition is due to Kantorovitch [1], p. 155, (α)–(β).

DEFINITION 7.4: *By a "direct component" of a vector lattice L, we mean the set L_i of $[0, \cdots, 0, f_i, 0, \cdots, 0]$ in a representation of L as a direct union $L_1 \times \cdots \times L_r$.*

THEOREM 7.11: *The direct components of any vector lattice are its* complemented normal subspaces.

Proof: The direct decompositions in the sense of linear spaces (ignoring order) correspond to choices of complementary subspaces—and the associated congruence relations preserve joins and meets if and only if the subspaces are normal.

137. Distributivity. Closely related to Theorem 7.6 is the following basic theorem on normal subspaces.

THEOREM 7.12: *The* normal *subspaces of any vector lattice form a distributive sublattice of the lattice of all subspaces.*

Proof: The intersection $S \frown T$ of two normal subspaces contains with any f all x such that $|x| \leqq |f|$ (since S and T do). Thus it is a normal subspace. Now for the

LEMMA: *If $x \leqq g + h$ $[x > 0, g > 0, h > 0]$, then $x = s + t$, where $0 \leqq s \leqq g$, $0 \leqq t \leqq h$.*

Proof: Set $s = x \frown g$; then $t = x - x \frown g = g \smile x - g \leqq (g + h) - g = h$. Obviously $0 \leqq s \leqq g$, $0 \leqq t$, $x = s + t$.

Now if $f = g + h$ $[g \in S, h \in T]$, and $|x| \leqq |f|$, then $x^+ \leqq |f| \leqq |g| + |h|$ can by the Lemma be written $s + t$, where $|s| \leqq |g|$, $|t| \leqq |h|$, whence $s \in S$, $t \in T$ if S and T are normal. Similarly, $-x^-$ is in $S + T$, whence so is $x = x^+ - (-x^-)$. Thus $S + T$ is a normal subspace, and normal subspaces are a sublattice of the lattice of all subspaces.

It remains to prove distributivity: that if S, T, and U are normal, then $S \frown (T + U) = (S \frown T) + (S \frown U)$. But since $S \frown (T + U) \geqq (S \frown T) + (S \frown U)$ in any case, and $x = x^+ - (-x^-)$, it suffices to show that every positive x in $S \frown (T + U)$ is in $(S \frown T) + (S \frown U)$. But $x \in S \frown (T + U)$ means that $x \in S$ *and* $x = t - u$ $[t \in T, u \in U]$. Hence $|x| \leqq |t| + |u|$, and (by the Lemma) $|x|$ is the sum of parts t^* of $|t|$ and u^* of $|u|$—whence $|x|$, and so x, is in $(S \frown T) + (S \frown U)$.

COROLLARY 1: *The direct components of any vector lattice form a Boolean algebra.*

COROLLARY 2: *Any two direct decompositions of a vector lattice have a common refinement, in the strict sense.*†

138. Examples. Function theory abounds in examples of normal subspaces of vector lattices. For instance, the set of functions equal to zero except on a finite set is a normal subspace in R^ω, R^c, and their subspaces. So is the set (B)

† We use the fact that either distributive law implies the other (Theorem 5.1), and also Theorem 5.15.

of bounded functions. Moreover, if (M) and (M^p) are understood as in §132, every (M^p) is a normal subspace of (M); so is every (l^p) a normal subspace of R^ω.

Again, the set N of functions vanishing except on a set of Lebesgue measure zero (so-called "null functions") is a normal* subspace of R^c and its subspaces; so is the subspace of functions vanishing except on a countable set.

Furthermore the "vector quotient-lattices" $(M)/N$ and $(M^p)/N$ define the usual "spaces" (S) of measurable functions and (L^p) as vector lattices. Another interesting space is the space $(M) \frown (B)/N$ of *bounded* measurable functions modulo null functions.†

139. Completeness of various vector lattices. Any direct union of (conditionally) σ-complete lattices or of complete lattices is itself σ-complete resp. complete. Hence the vector lattices R, R^n, R^ω, R^c are complete. Moreover since normal subspaces are convex,

THEOREM 7.13: *Any normal subspace of a complete (σ-complete) vector lattice is complete (σ-complete).*

It is a corollary that the spaces (B), (b), (l^p) and N are complete vector lattices.

Now consider the space (S) of measurable functions $f(x)$, modulo null functions. Let $X(f, a)$ denote the set on which $f(x) \geqq a$. Then $X(f, a)$ is an order-preserving function from the interval R: $(-\infty, +\infty)$ to the complete Boolean algebra M/N discussed in §123. Moreover if $X(f, a) = X(g, a)$ for all a, then the set on which $| f(x) - g(x) | < 1/n$ is null for all n; hence $f(x) - g(x)$ is a null function and $f = g$. Again, for any $f \in (S)$, $\inf_a X(f, a) = O$ and $\sup_a X(f, a) = I$. Conversely, any $X(a)$ having the properties described is an $X(f, a)$ —and $f(x)$ can be constructed through approximating step-functions having countable values. Since, finally, $f \geqq g$ is equivalent to $X(f, a) \geqq X(g, a)$ for all a, we conclude the

LEMMA: *The space (S) is lattice-isomorphic with the set of functions from R to M/N, which preserve order and satisfy* $\inf_a X(a) = O$ *and* $\sup_a X(a) = I$.

But this is a convex subset of the lattice $(M/N)^R$, and the latter is complete since M/N is. We conclude

THEOREM 7.14: *The space (S) of measurable functions modulo null functions is a complete vector lattice.*

It is a corollary that the spaces (L^p) and $(S) \frown (B)$ are complete, being normal subspaces of the space (S).

Again, by Theorem 2.2, the semi-vector lattices (SH) and (Con) of §133 are

* It is even σ-normal, in the sense that it contains with any countable set $\{x_i\}$, an upper bound to the set. Incidentally, the set of functions defined on $(-\infty, +\infty)$ and vanishing except on a set of finite measure forms a normal subspace of the set of all functions defined on $(-\infty, +\infty)$.

† A curious fact is that (L^2) and (l^2) are not order-isomorphic. For there exist in (l^2) positive elements, whose positive subelements form a chain; none such exist in (L^2).

complete. The space (USC) is also complete: given any family of $u_\alpha(x)$ in the space, set $f(x) = \sup u_\alpha(x)$ and $u(x) = \limsup_{t\to x} f(t)$; then $u(x)$ will be a least upper bound to the $u_\alpha(x)$ in the space (USC). Dually, the space (LSC) is (conditionally) complete. On the other hand, (C) is not even σ-complete: set $u_n(x) = 0$ on $(-\infty, 0)$, $u_n(x) = nx$ on $(0, 1/n)$, and $u_n(x) = 1$ on $(1/n, +\infty)$.

140. Order topology. The reader may recall from Chapter II, definitions of lim sup, lim inf, (o)-convergence, and star-convergence in general σ-lattices. We shall now consider special properties possessed by the resulting topologies, in the case of *vector* lattices.*

Lemma 1: *If $f_n \downarrow 0$ and $g_n \downarrow 0$, then $(f_n + g_n) \downarrow 0$.*

Proof: Since $\{f_n\}$ and $\{g_n\}$ are decreasing, clearly

$$\inf \{f_n + g_n\} = \inf_{m,n} \{f_m + g_n\} = \inf_m \{\inf_n (f_m + g_n)\}.$$

But since $f \to f + a$ is a lattice automorphism,

$$\inf_m \{\inf_n (f_m + g_n)\} = \inf_m \{f_m + \inf_n g_n\} = \inf_m \{f_m\} = 0.$$

Since $\{f_n + g_n\}$ is decreasing, this implies $\{f_n + g_n\} \downarrow 0$.

Incidentally, this argument can be generalized to show that an order-preserving function of two lattice variables, which is continuous in each variable separately, is continuous in the product-lattice of the two variables.

Lemma 2: *The sequence $\{f_n\}$ (o)-converges to f if and only if $|f_n - f| \leqq w_n$ for some $w_n \downarrow 0$.*

Proof: Since $x \to x + f$ is a lattice automorphism which leaves absolutes of differences unchanged, we can assume $f = 0$. But if f_n satisfies $u_n \leqq f_n \leqq v_n$ for some $u_n \uparrow 0$ and $v_n \downarrow 0$, then $|f_n| \leqq v_n + (-u_n) = w_n$, where $w_n \downarrow 0$ by Lemma 1. Conversely, if $|f_n| \leqq w_n$ and $w_n \downarrow 0$, then $-w_n \leqq f_n \leqq w_n$, where $-w_n \uparrow 0$ by Theorem 7.1.

It is a corollary that in a σ-complete vector lattice we have the following "generalized Cauchy condition": $\{f_n\}$ is convergent if and only if $|f_m - f_n| \to 0$ when $m, n \to \infty$.

Theorem 7.15: *With respect to its order topology, any σ-complete vector lattice is a topological linear space:† the functions $f + g$, λf, $f \smile g$, and $f \frown g$ are continuous.*

* The properties of (o)-convergence given in the present section are taken from Kantorovitch [1], Thms. 10–21.

† Cf. Theorem 6.9. In the sense of J. von Neumann, *Topological linear spaces*, Trans. Am. Math. Soc., 37 (1935), 1–19. This means it is a topological algebra in the general sense of van Dantzig. To show that Theorem 7.15 is not trivial note that it does not hold in the semi-vector lattice (USC). Set $a(x) = 0$ $[x \neq 1/2]$, $a(1/2) = -1/2$, $u_n(x) = 0$ $[2^n x$ not an integer or $x = 1/2]$, and $u_n(x) = -1/2 + |x - 1/2|$ otherwise. Then $u_n \downarrow u$, where $u(x) = -1/2 + |x - 1/2|$, $u_n \smile a = 0$, and yet $u \smile a = a < 0$.

Proof: If $\lambda_n \to \lambda$, $f_n \to f$, and $g_n \to g$, then by Lemma 1, $|(f_n + g_n) - (f + g)| \leqq |f_n - f| + |g_n - g| \to 0$. That is, $f_n + g_n \to f + g$. Using Theorem 7.8 and Lemmas 1–2, we see by similar formulas that $f_n \smile g_n \to f \smile g$ and $f_n \frown g_n \to f \frown g$. Finally,

$$|\lambda_n f_n - \lambda f| \leqq (\sup |\lambda_n|) \cdot |f_n - f| + |\lambda_n - \lambda| \cdot |f|$$

and this (o)-converges to zero by Lemma 1.

141. Star topology. But it is easy to show that in any L-space, functions continuous with respect to the L-topology are ipso facto continuous in the L^*-topology (one need only take successive subsequences from arbitrary sequences). Hence Theorem 7.15 also holds for star-convergence.

We shall now prove that, in the space (S), star-convergence is equivalent to "convergence in measure," in the usual sense.† Indeed, suppose $\{u_n\}$ star-converges to 0 in the space (S) of measurable functions with domain $[0, 1]$, modulo null functions. If the set X'' on which $|u_n(x)| \geqq \epsilon$ is of measure exceeding ϵ for an infinite set of n, then there exists a subsequence for which this is true identically, and no subsequence of this can (o)-converge to zero, contrary to hypothesis. Hence for sufficiently large n, $|u_n(x)| \geqq \epsilon$ on a set of measure at most ϵ, and star-convergence implies "convergence in measure." To see the converse, note that it suffices to get a subsequence (o)-converging. But $\sum_{k=N}^{\infty} |u_{n(k)}(x)| \downarrow 0$ if $\epsilon(k) < 2^{-k}$.

We shall also verify below that star-convergence is equivalent to the metric convergence of Banach with L_p, and in many other Banach spaces.

142. Relative uniform convergence. Following a basic idea of E. H. Moore,‡ we shall say that a sequence $\{f_n\}$ of elements of a vector lattice converges "relatively uniformly" to an element f, if and only if for some u and $\lambda_n \downarrow 0$, $|f_n - f| \leqq \lambda_n u$.

The relative uniform topology thereby defined can be correlated with the order-topology through

THEOREM 7.16: *In any σ-complete vector lattice, relative uniform convergence implies (o)-convergence. The reverse implication holds, provided $u_n \downarrow 0$ implies that some sequence $ku_{n(k)}$ is bounded.*

Proof: If $|f_n - f| \leqq \lambda_n u$ and $\lambda_n \downarrow 0$, then $\{f_n\}$ (o)-converges to f by Lemma 2 of §140 and Theorem 7.3. Conversely, if $|f_n - f| \leqq u_n$, where $u_n \downarrow 0$, and if u is an upper bound to $ku_{n(k)}$, then $|f_n - f| \leqq u/k$ for all $n > n(k)$.

The above hypothesis holds in the "regular" case of Kantorovitch.§ It holds

† Cf. Kantorovitch, Comptes Rendus, 201 (1935), p. 1457. Also, (o)-convergence is convergence almost everywhere! Using Theorem 7.16, one sees that it is also relative uniform convergence.

‡ *The New Haven Mathematical Colloquium*, New Haven, 1914, pp. 31, 39. Cf. also Bull. Am. Math. Soc., 18 (1912), p. 334. A similar definition is given by Kantorovitch [1], p. 142.

§ [1], cf. Thm. 26⁺. Kantorovitch in his original paper (Doklady) used a weaker definition: that if $\lambda_n \downarrow 0$ implies $\lambda_n u_n \to 0$, then $\{u_n\}$ was bounded. This holds (apparently) in the spaces (B), (b).

in the space (S), since $| u_n(x) | < 1/k^2$ except on a set of measure at most $1/k^2$, for all $n > N(k)$, provided $N(k)$ is suitably chosen. Similarly, it holds in the space (L^p). It does not hold in the space (b) of bounded sequences† or in (B).

143. Relative uniform star-convergence. With respect to relative uniform convergence, any vector lattice forms an L-space. But derived sets need not be closed, and it need not form an L^*-space. But as we know,‡ the latter defect can be eliminated by defining relative uniform star-convergence to mean relative uniform convergence of some subsubsequence of every subsequence.

We shall show below (§148) that in the case of Banach spaces, the resulting relative uniform star-topology is equivalent to the metric topology; hence it makes derived sets automatically closed.

144. Additive functions between vector lattices. Consider the functions T: $f \to fT$ from a vector lattice F to a complete vector lattice X. We shall restrict our attention to "additive" functions—to functions satisfying the identity $(f + g)T = fT + gT$.

It is well-known that the additive functions between any two linear spaces themselves form a linear space, if one defines λT and $T + U$ through the identities $f(\lambda T) = \lambda(fT)$ and $f(T + U) = fT + fU$. In the present case, we can further partially order this linear space, through

Definition 7.5: *We shall call T "non-negative" if and only if*

M2: *$f \geqq g$ implies $fT \geqq gT$,*

in words, when T preserves order. This is equivalent to requiring that $f - g \geqq 0$ imply $(f - g)T \geqq 0$ —in words, that T preserve non-negativeness.

Theorem 7.17: *Under Definition 7.5, the additive functions from F to X form a partially ordered linear space, to be denoted§ (X^F).*

The proof is trivial. Since X is a partially ordered linear space, if $f \geqq 0$ implies $fT \geqq 0$ and $fU \geqq 0$, then it implies $f(T + U) \geqq 0$ and $f(\lambda T) \geqq 0$ for all $\lambda \geqq 0$, proving F3 and F1. Also, if $f \geqq 0$ implies $fT \geqq 0$ and $f(-T) \geqq 0$, then it implies $fT = 0$ —whence $gT = g^+T - (-g^-)T = 0$ for all g, proving F2.

On the other hand, they do not form a vector lattice, even in the simplest case.|| We shall now show that in order to get a vector lattice, we have to restrict ourselves to what may properly be called "bounded" additive functions.

† For let u_n be the sequence composed of n zeros followed by all ones. Then $u_n \downarrow 0$, yet irrespective of $n(k)$, $\{ku_{n(k)}\}$ is unbounded. In fact, the sequence (o)-converges to 0 without converging to 0 relatively uniformly.

‡ Cf. Urysohn, op. cit., in footnote ‡, p. 30. The first defect is considered by Moore, op. cit., p. 40.

§ One can of course also define (linear) sums $F \oplus G$ of vector lattices in an obvious way, leading to an intriguing calculus. This must not be confused with the calculus of §16.

|| In case F is infinite-dimensional, there exist unbounded linear functionals. Using a Hamel basis (i.e., regarding R as an infinite-dimensional linear space with rational scalars) one can even construct unbounded additive functionals on R.

145. Bounded additive functions. Indeed, if an additive function T from F to X lies in any vector lattice, then the set $\{T, 0\}$ must be bounded—i.e., U and V must exist, such that $V \leqq 0 \leqq U$, $V \leqq T \leqq U$.

DEFINITION 7.6: *An additive function T will be called "bounded" if and only if the set $\{T, 0\}$ is bounded.*

LEMMA 1: *If T is bounded, then it carries bounded sets (sets H satisfying $a \leqq H \leqq b$) into bounded sets.*

Proof: If $a \leqq h \leqq b$, and $V \leqq T \leqq U$, $V \leqq 0 \leqq U$, then

$$hT = aT + (h - a)T \leqq aT + (h - a)U \leqq aT + (b - a)U,$$

and dually, $hT \geqq bT + (b - a)V$. Hence the set of hT $[h \epsilon H]$ is bounded.

LEMMA 2: *If T carries bounded sets into bounded sets, then $T^+ = T \smile 0$ exists. In fact, $fT^+ = \sup_{0 \leqq x \leqq f} xT$ iff $f \geqq 0$, and $fT^+ = f^+T^+ - (-f)T^+$ otherwise.*

Proof: Firstly, T^+ is additive. Indeed, by the Lemma of §137, sup xT for $0 \leqq x \leqq f + g$ is sup $(y + z)T$ = sup $(yT + zT)$ for $0 \leqq y \leqq f$, $0 \leqq z \leqq g$. And this is (sup yT) + (sup zT), since on the one hand the latter is an upper bound to the $yT + zT$, and on the other any upper bound to the $yT + zT$ contains yT + sup zT for every fixed y, and hence (also by Theorem 7.1) sup yT + sup zT.

It remains to show that $T^+ = T \smile 0$. But clearly $f \geqq 0$ implies $f(T^+ - T) \geqq fT - fT = 0$ and $f(T^+ - 0) \geqq 0T - f0 = 0$; hence T^+ is an upper bound to 0 and T. Conversely, if U is any (additive) upper bound, then for all x between 0 and f,

$$fU = xU + (f - x)U \geqq xT + (f - x)0 = xT$$

whence $f(U - T^+) = fU - fT^+ \geqq 0$ for all $f \geqq 0$, and $u \geqq T^+$.

THEOREM 7.18:† *The bounded additive functions form a vector lattice X^F. This is embedded in the (X^F) of Theorem 7.17, and contains every vector lattice so embedded.*

Proof: If $V \leqq 0$, $T \leqq U$ and $V^* \leqq 0$, $T^* \leqq U^*$, then clearly $V + V^* \leqq 0$, $T + T^* \leqq U + U^*$ and $V \leqq 0$, $\lambda T \leqq \lambda U$ or the reverse for all λ; hence the bounded additive functions form a subspace of (X^F). By Theorem 7.2 and Lemma 2, this subspace is a vector lattice. And by the remark preceding Definition 7.6, every vector lattice embedded in (X^F) is contained in it.

THEOREM 7.19: *Each of the following conditions is equivalent to boundedness, (1) T carries bounded sets into bounded sets, (2) T is the difference of non-negative functions.*

† The theory of §§143-5 is due to the author [8]. Cf. *On dependent probabilities and spaces (L)*, Proc. Nat. Acad. Sci., 24 (1938), 155-9, and Abstract 43-3-21 of the Bulletin of the American Mathematical Society; also F. Riesz [2].

Proof: By Lemma 1, boundedness implies (1); by Theorems 7.18 and 7.5, (1) implies (2). Finally, if $T \geqq 0$, $U \geqq 0$, then $-T - U \leqq T - U \leqq T + U$; hence (2) implies boundedness.

146. Functionals and conjugate spaces. The case that X is the real number system—i.e., of real-valued functions or "functionals" on a vector lattice—has attracted especial attention from mathematicians since the beginning.

In this connection, we may note that as regards such functionals, the ideas used above already appeared in §50. Indeed, the whole theory of modular functionals and their variation extends to functions with values in a complete vector lattice, and §§145–6 are largely devoted to this extension.

Thus to start with, any additive functional is modular. Indeed, by the Corollary to Theorem 7.5, it satisfies

$$\text{M1: } fT + gT = (f + g)T = (f \frown g + f \smile g)T = (f \frown g)T + (f \smile g)T.$$

Again, an additive functional is "bounded," if and only if it is of bounded variation on every bounded chain. Moreover when $f \geqq 0$, fT^+ and fT^- are the positive and negative variations of T on the interval between 0 and f.

We shall now start a new line of thought, by defining the "conjugate space" F^* of a linear lattice F, as the vector lattice of bounded additive functionals on F.

We shall show later that our definition specializes to Banach's concept of a "conjugate space," although apparently quite different from it. Surprisingly, it includes as another special case, the notion of a "Dualraum" introduced by Koethe and Toeplitz.†

Without attempting profundities, we shall prove the important relation $F \leqq (F^*)^*$. Indeed, every $f \in F$ defines an additive functional ϕ on F^*: $\phi(T) = fT$. Moreover every such ϕ is bounded; since $f = f^+ - (-f^-)$, we need only prove this when $f \geqq 0$. But if $f \geqq 0$ and $V \leqq T \leqq U$, then $fV \leqq fT \leqq fU$; and it follows that the $\phi(T)$ on any bounded set (between any V and U) are bounded.

147. Metric vector lattices. We shall now reintroduce metric considerations, through

Definition 7.7: *By a Banach lattice, will be meant a vector lattice which is a Banach space‡ with norm* $|f|$, *in which* $|f| \leqq |g|$ *implies* $|f| \leqq |g|$.

Examples: Every example of a Banach space cited by Banach (op. cit.) is a Banach lattice in the sense of Definition 7.7.

† *Lineare Räume* ⋯ , Jour. of Math., 171 (1934), 193–226. At least, if we assume what is true in all the examples cited by them: that F contains every $(0, \cdots, 0, 1, 0, \cdots)$, and $|x|$ with x. For in this case, every positive functional assumes some non-negative value λ_k on x_k, and sup $\Sigma\, \lambda_k\xi_k$ on $\Sigma\, \xi_k x_k$. Hence it, and hence every difference of positive functionals, yields an absolutely convergent $\Sigma\, \lambda_k \mid \xi_k \mid$, and so is in Koethe's Dualraum. Conversely, if u is in this Dualraum, then the positive and negative u_k define new monotone additive functionals, whose sum is u.

‡ Or B-space in the sense of Banach [1], p. 52. The condition prescribed implies that $|f| = \|f\|$ for all f.

LEMMA: *No vector lattice L is a Banach lattice which possesses a subset $\{f_n\}$, such that all sequences $\{\lambda_n f_n\}$ are bounded.*

Proof: Set $\lambda_n = n/|f_n|$. If u were an upper bound to all $\lambda_n f_n$, then $|u|$ would have to exceed every n, since $|\lambda_n f_n| = n$. This is impossible.

In the spaces (S) and R^c, the functions $f_n(x) = 1$ on $(0, 1/n)$ and zero elsewhere have this property. On the other hand, the norm $\int |f(x)|/[1 + |f(x)|]\, dx$ in the space (S) satisfies all conditions except $|\lambda f| = |\lambda|\cdot|f|$.

148. Their metric topology. It is well-known that if one defines the "distance" between elements f and g of a Banach space as $|f - g|$, then one gets a metric space. Relative to this distance,

THEOREM 7.20: *The operations $f + g$, $f \frown g$, and $f \smile g$ are metrically uniformly continuous (of modulus unity).*

Proof: By commutativity, we need only prove uniform continuity in f. But since

$$|(f + g) - (f^* + g)| = |f - f^*|,$$

this is true of $f + g$. To prove it for $f \frown g$ and $f \smile g$, we need only use Theorem 7.8 (which bounds $|f \frown g - f^* \frown g|$ and $|f \smile g - f^* \smile g|$ by $|f - f^*|$), and observe that in consequence $|f - f^*|$ bounds $|f \frown g - f^* \frown g|$ and its dual.

It is a corollary that if $\sum_{n=1}^{\infty} a_n = a$, $\sum_{n=1}^{\infty} b_n = b$, $\sum_{n=1}^{\infty} x_n = x$ metrically, and $a_n \leqq x_n \leqq b_n$ for all n, then $a \leqq x \leqq b$.

THEOREM 7.21: *Metric convergence is equivalent to relative uniform star-convergence, in any Banach lattice.*

Proof: By the homogeneity of both topologies, we need only consider convergence to 0. But if $|f_n| \leqq \lambda_n u$ and $\lambda_n \downarrow 0$, then clearly $|f_n| \leqq |\lambda_n u| = |\lambda_n|\cdot|u| \downarrow 0$. Thus relative uniform star-convergence implies metric star-convergence and so metric convergence. Conversely, if $|f_n| \to 0$, we can so choose $n(k)$ that $k^3 |f_{n(k)}| \to 0$, and then construct $\bar{u} = \sum_{k=1}^{\infty} k |f_{n(k)}|$ with $|f_{n(k)}| \leqq \bar{u}/k$.

By combining Theorem 7.21 with Theorem 7.16, we obtain the relation between metric convergence and star-convergence.†

149. The two kinds of boundedness. Since $a \leqq x \leqq b$ implies $|x| \leqq |a| + |b - a|$, order-boundedness (§36) of a subset of a Banach lattice clearly implies metric boundedness. The converse is not however usually true. Thus in (L^p) and (l^p), there is no upper bound to the elements of norm one—although in the spaces R^n, (B) and (b) there is.

But paradoxically, we have the remarkable result

† Thus (o)-convergence, even of monotone sequences, need not imply metric convergence (footnote on page 114). Again, in the "regular" case of Kantorovitch, metric convergence and star-convergence are equivalent. Incidentally, for monotone sequences, metric convergence implies (o)-convergence; in any case, metric convergence implies star-convergence.

THEOREM 7.22: *Metrical "boundedness"* and the "boundedness" of Definition 7.6 are equivalent for additive functionals T on a Banach lattice.*

Proof: If T is bounded metrically, then it is bounded on any interval $a \leqq x \leqq b$; hence it is bounded in the sense of Definition 7.6. If T is metrically unbounded, then a sequence $\{x_n\}$ exists with $|x_n| \leqq 2^{-n}$ yet $|x_nT| \uparrow +\infty$. Hence (by the Corollary to Theorem 7.20) the elements $a = \sum_{n=1}^{\infty} x_n^-$ and $b = \sum_{n=1}^{\infty} x_n^+$ bound a set of y on which $|yT|$ is unbounded; that is, T is unbounded in the sense of Definition 7.6.

150. Strictly monotone norm. We shall call the norm in a Banach lattice "strictly monotone" when, given $\epsilon > 0$, one can find $\delta > 0$ so small that if $f \geqq 0$, $g \geqq 0$, and $|f| \leqq 1$, then $|f + g| \leqq |f| + \delta$ implies $|g| \leqq \epsilon$.

Examples: Since $|f + g| = |f| + |g|$, the space (L) obviously has strictly monotone norm. Again, when $f > 0$, $g > 0$, $|f(x) + g(x)|^p \geqq |f(x)|^p + |g(x)|^p$ for all x. Hence in (L^p) and (l^p) $|f + g|^p \geqq |f|^p + |g|^p$, whence we infer that $|f + g| \leqq (|f|^p + \epsilon)^{1/p}$ implies $|g| \leqq \epsilon^{1/p}$ —i.e., that the norm is strictly monotone.

On the other hand, the norm in (B), (b) and (C) is not strictly monotone.

In what follows, we shall assume that we are dealing with an "SMB-lattice," that is, a Banach lattice in which the norm is strictly monotone.

THEOREM 7.23: *In an SMB-lattice, any metrically bounded set of elements with the property of Moore-Smith converges metrically.*

Proof: Since $f \to f + a$ is an isometric lattice-automorphism, we can assume the elements f_α of the set are non-negative. Again, by changing the scale, we can assume $|f_\alpha| \leqq 1$ for all α. But in this case, if α is so chosen that $|f_\alpha| \geqq \sup |f_\alpha| - \delta$, we will have $|f_\beta - f_\alpha| \leqq \epsilon$ for all $f_\beta > f_\alpha$, proving our result.

COROLLARY: *Any SMB-lattice is (conditionally) complete.*

For if a set has an upper bound, so do the joins of the finite subsets of the set. But these are metrically bounded and have the property of Moore-Smith; hence we can apply Theorem 7.23.

THEOREM 7.24: *In an SMB-lattice, (o)-convergence and relative uniform convergence are equivalent.*

Proof: A monotone sequence automatically has the Moore-Smith property. If (o)-convergent, it is also bounded, and so converges metrically (Theorem 7.23), and so (last part of proof of Theorem 7.21) relatively uniformly. But in any σ-complete Banach lattice, by Theorem 7.16, relative uniform convergence implies (o)-convergence.

* We shall call (Banach [1], p. 53) an additive function T between Banach spaces metrically bounded when it satisfies one of the three equivalent conditions: (1) it transforms metrically bounded sets into (metrically) bounded sets, (2) it is continuous in the metric topology, (3) for some $\|T\|$, $|fT| \leqq \|T\| \, |f|$ for all f.

It is a corollary that star-convergence and metric convergence are equivalent.*

Finally, we may note that Theorem 7.22 extends to functions T with values in any SMB-lattice. Indeed, the second half of the proof need not be changed. As regards the first half, note that since any two decompositions of the interval $0 \leqq x \leqq f$ have a common refinement, the xT $[0 \leqq x \leqq f]$ have the property of Moore-Smith. Hence if T is metrically bounded, so are the xT, and by Theorem 7.23 they converge to a supremum fT^+. Dually, the xT have a lower bound, and so T is bounded in the sense of Definition 7.6.

151. A decomposition theorem. Let L be any SMB-lattice, and $\lambda(f)$ any bounded additive functional on L. We shall show that $\lambda(f)$ decomposes L into components on which $\lambda(f)$ is positive, negative, and zero respectively. First

LEMMA 1: *Let N^+, N^- and N^0 be defined as the sets of f such that $0 < x \leqq |f|$ implies $\lambda(x) > 0$, $\lambda(x) < 0$, and $\lambda(x) = 0$, respectively.* Then *N^+, N^-, and N^0 are independent normal subspaces.*

Proof: Suppose $f \in N^+$ and $g \in N^+$. Then $0 < x \leqq |f + g|$ implies $0 < x \leqq |f| + |g|$, whence $x = y + z$ $[0 \leqq y \leqq |f|, 0 \leqq z \leqq |g|$, and $y > 0$ or $z > 0]$, and $\lambda(x) = \lambda(y) + \lambda(z) > 0$. That is, N^+ is a subspace. Again, if $f \in N$ and $|h| \leqq |f|$, then $0 < x \leqq |h|$ implies $0 < x \leqq |f|$, whence $\lambda(x) > 0$ and $h \in N^+$. We conclude: N^+ is a normal subspace. Similarly, N^- and N^0 are normal subspaces.

Now using the distributive law, independence follows from pairwise independence—from $N^+ \frown N^- = N^- \frown N^0 = N^0 \frown N^+ = 0$. And this is true if the pairwise intersections have no positive elements. But this is trivial: if $f > 0$, then $f \in N^+$ implies $\lambda(f) > 0$, $f \in N^-$ implies $\lambda(f) < 0$, and $f \in N^0$ implies $\lambda(f) < 0$; three mutually exclusive conditions.

THEOREM 7.25: *L is the direct union of N^+, N^-, and N^0.*

Proof: By Theorem 7.11, Lemma 1 and the relation $N^+ + N^- + N^0 = L$ would imply Theorem 7.25. Hence it suffices to prove this relation—for which strict monotonicity is really needed.† To prove it, we require

LEMMA 2: *The functional $\lambda(x)$ attains its supremum M on any interval* $0 \leqq x \leqq f$.

Proof: Choose x_i so that $\lambda(x_i) > M - 3^{-i}$; we shall show that $g = \lim\sup x_i$ —which exists by conditional completeness of L —satisfies $\lambda(g) = M$. Indeed,

$$\lambda(x_i \smile x_j) = \lambda(x_i) + \lambda(x_j) - \lambda(x_i \frown x_j) > M - 3^{-i} - 3^{-j},$$

whence $\lambda(x_i \smile x_{i+1} \smile \cdots \smile x_{i+n}) > M - (3^{-i} + \cdots + 3^{-i-n})$ exceeds $M - 2 \cdot 3^{-i}$. Passing to the limit once, $M \geqq \lambda(\sup_{k \geqq i}\{x_k\}) \geqq M - 2 \cdot 3^{-i}$. Passing to the limit similarly again, by the continuity in the metric topology,

* This is closely related to results of Kantorovitch (Comptes Rendus, 201 (1935), p. 1457, and [1], p. 154. The case $p = 1$ is covered by Theorem 3.13.

† It is not needed for Lemma 1, which holds in any vector lattice. Lemmas 2 and 3 do not hold in the space (C), nor does Theorem 7.25.

which (Theorem 7.24) implies continuity in the (o)-topology, we get $\lambda(y) = M$.
The relation $L = N^+ + N^0 + N^-$ is now obvious from

Lemma 3: *In Lemma 2, let u be the infimum of the x between 0 and f such that $\lambda(x) = M$; define v dually; let $w = f - u - v$. Then $u \in N^+$, $v \in N^-$, $w \in N^0$, and $f = u + v + w$.*

Proof: The existence of u and v (and thus of w) follows from the conditional completeness of L. Again, if $\lambda(x) = \lambda(y) = M$, then $\lambda(x \frown y) + \lambda(x \smile y) = 2M$, whence $\lambda(x \frown y) = \lambda(x \smile y) = M$. It follows by Theorem 7.23 and continuity that $\lambda(u) = M$. Moreover $0 < x \leqq u$ implies $\lambda(u - x) < \lambda(u)$ and so $\lambda(x) = \lambda(u) - \lambda(u - x) > 0$; hence $u \in N^+$. Dually, $v \in N^-$. Hence $u \frown v = 0$ by Lemma 1, whence $u + v = u \smile v \leqq f$ and $0 \leqq w \leqq f$. Finally, $0 < x \leqq w$ implies $\lambda(x) + \lambda(u) = \lambda(x + u) \leqq \lambda(u)$, whence $\lambda(x) \leqq 0$. Dually, it implies $\lambda(x) \geqq 0$, and so $\lambda(x) = 0$. Hence $w \in N^0$. But $f = u + v + w$ is obvious, completing the proof.

Theorem 7.25 does not hold in general Banach lattices.

152. Finite-dimensional case. Let L be any vector lattice of finite dimensions n. The elements $f \geqq 0$ in L form a convex cone P whose vertex is 0. Moreover since $f = f^+ - (-f)^+$, P has an interior: it is n-dimensional.

Lemma 1: *If L is complete or a Banach lattice, then the cone P is closed in the Cartesian topology.*

Proof: If L is a Banach lattice, then the solutions of $f \frown 0 = 0$ form a closed set in the (metric) Cartesian topology, by Theorem 7.20. Again, unless P is closed, letting a be an element on the boundary of P not belonging to P, the upper bounds to the $-a/n$ include the interior of P and a part of its boundary, but not 0. Hence sup $\{-a/n\}$ does not exist, and L is not (conditionally) σ-complete.

Remark: It seems likely that the results of this section hold except in the pathological non-Archimedean case that for some f the set $\{f, 2f, 3f, \cdots\}$ is bounded. In fact, the author conjectures that every finite-dimensional vector lattice is a mixed direct and lexicographic union of R. Cf. the end of §130 for illustration of the possibilities.

Lemma 2: *If the cone P is closed in the Cartesian topology, then the conclusions of Theorem 7.25 hold.*

Proof: Lemma 1 of §151 holds for any linear functional on any vector lattice. Lemma 2 holds since any linear functional* is continuous in the Cartesian topology, while the interval between 0 and f, being bounded and closed, is compact in the Cartesian topology. Finally, Lemma 3 holds since the solutions of $\lambda(x) = M$, $0 \leqq x \leqq f$, form a closed, convex (in the lattice sense) set, whose infimum provides the desired u. Theorem 7.25 is a corollary of these three lemmas.

* I.e., additive functional satisfying $\lambda(\alpha x) = \alpha \cdot \lambda(x)$ for all α.

THEOREM 7.26: *If an n-dimensional vector lattice is complete or a Banach lattice, then it is isomorphic with R^n.*

Proof: Pass an $(n - 1)$-dimensional plane through 0 and any point of the interior of the cone P. Construct $\lambda(x)$ so as to vanish on this plane; then N^+, N^- and N^0 will be direct components of L of fewer dimensions. The conclusion now follows by induction on the dimension-number.*

* Additional literature: L. Kantorovitch, *The method of successive approximation for functional equations*, Acta Math., 71 (1939), 63-97.

Incidentally, the following conditions on a normal subspace N of a complete vector lattice are equivalent: (1) N is complemented, (2) N contains with any bounded set its l.u.b., (3) N is closed in its (o)-topology under Moore-Smith limits.

CHAPTER VIII

APPLICATIONS TO LOGIC

153. Introduction. In probability and mathematical logic, one is constantly confronted with the formal systems considered in the earlier chapters of this book. Thus traditional logic involves Boolean algebras, and the modifications of it proposed by various writers involve other lattices. Again, mathematical probability deals with modular functionals on Boolean algebras—and so with "Banach lattices."

These superficial observations suggest the following program. First, the fundamental ideas of logic and probability must be formulated, whenever possible, in terms of lattice theory. This accomplished, the significance for logic and probability of the main results of lattice theory must be pointed out. And finally, existing techniques must be supplemented by lattice-theoretic methods, to help solve outstanding problems and to suggest new ones.

This program is not altogether original. The first part of it was in Boole's mind,* and has been accomplished by his followers. More recently, Tarski has discussed the significance for logic of technical theorems on Boolean algebra—and thus may be said to have undertaken the second part of the program.

Perhaps the most original development is the new theory of dependent probabilities given below, including a new ergodic theorem. This distinctly belongs to the third part of the program.

The reader should be cautioned that the author has not tried to give a complete survey of mathematical logic. This difficult and paradoxical field obviously lies outside the scope of a single chapter, and attention has been systematically confined to its *lattice-theoretic* aspects.

154. Algebra of attributes. In logic, the concept of an object having "attributes," "properties," or "qualities" (these terms are synonymous) is absolutely fundamental.†

Attributes may be designated by adjectives (viz., red, liquid, dead, etc.) or by generic nouns (viz., beast, tree, ocean, etc.). From the point of view of logic, adjectives and generic nouns are equivalent—to say "water is liquid" is the same as to say "water is a liquid."

One can *combine* attributes by use of the two conjunctions *and, or* (viz., red

* Boole's design ([1], p. 1) was "to establish the science of Logic and construct its method" upon "the symbolic language of a Calculus," and "to make that method itself the basis of a general method for the application of the mathematical doctrine of Probabilities." The first part of this design was already contemplated by Leibniz!

† Indeed, Boole ([1], p. 27, Prop. I) asserts that all reasoning can be reduced to the discussion of objects and their attributes. His identification of attributes with the classes of objects possessing them (cf. §155) is made on pages 28 and 43.

and liquid, red or liquid); one can form from any attribute its negative by using the adverb *not* (viz., not red). One can also relate attributes by inclusion; thus the attribute of being an ocean includes the attribute of being liquid ("every ocean is liquid").

THEOREM 8.1: *Let attributes be designated by letters, "and," "or," "not" by the symbols $\smile$, $\frown$, $'$ respectively, and "every x is y" by "$x \geqq y$." Then attributes constitute a Boolean algebra.*

This proposition can be verified by inspection of L1–L7 and illustration through examples. It is however so well-known, and has been discussed in such detail by so many authors, including Boole himself, that we shall assume it.*

It is a corollary of Theorem 8.1 that Boolean algebra is an "algebra of logic," and that every theorem on Boolean algebra (cf. Chapter VI) can be interpreted as a theorem on logic.

In particular, we can infer: (1) "or" and "not" can be defined in terms of "and," (2) at most 2^{2^n} different attributes can be constructed from n given ones, by use of "and," "or," "not," (3) every true identity in the calculus of attributes is demonstrable from L1–L7; every identity not demonstrable from L1–L7 implies $O = x = I$ for all x, when added to them.

Proof of (1): $x \geqq y$ is the same as $x = x \smile y$, while "meet" and "complement" can be defined in terms of inclusion. Proof of (2): Theorem 6.8. Proof of (3): Substitutions of x_i' for x_i—which can be validly made in *identities*—are transitive in the group-theoretic sense on the points (prime quotients) of the free Boolean algebra generated by n symbols. Hence if one point is annulled, all are.

We can interpret the second part of (3) as follows: the classical logic of attributes cannot be *strengthened* without giving rise to absurdities; it can only be *weakened*.

155. Boole's dual isomorphism. It is natural to identify each attribute x with the class $\hat{x}$ of all objects (or "things") possessing that attribute. Moreover one can prove

THEOREM 8.2: *The correspondence $x \rightarrow \hat{x}$ is a dual isomorphism. Thus $\widehat{x \smile y} = \hat{x} \frown \hat{y}$, $\widehat{x \frown y} = \hat{x} \smile \hat{y}$, $(\widehat{x'}) = (\hat{x})'$.*

As regards the three identities, the set of objects having attribute x *and* attribute y is (by definition of set-product) the product of the set of objects having x and that of those having y. The other two identities follow similarly from the definitions of set-union and set-complement.

Thus the correspondence is a dual homomorphism. Hence if its inverse exists and is single-valued, it is a dual isomorphism. But with each class X of objects one can associate the attribute $a(X)$ of "membership in X," in a single-valued fashion. Moreover $\hat{a}(X) = X$ for all X, and $a(\hat{x}) = x$ for all x; thus, to be an

* Biology furnishes excellent illustrations. Thus use mammal, vertebrate, carnivorous, cannibal, quadruped, etc., as attributes.

equilateral triangle is the same as to be a member of the class of equilateral triangles. That is, the correspondences are inverse, completing the proof.

The algebra of attributes is often called the "logic of intension," and the algebra of classes, the "logic of extension." Theorem 8.2 states that the two are dually isomorphic. But any Boolean algebra is dually isomorphic with itself, giving the corollary: the logic of intension and the logic of extension are isomorphic.

Theorem 8.2 also corroborates Theorem 8.1, since the subsets of any aggregate form a Boolean algebra, and anything dually isomorphic to a Boolean algebra is a Boolean algebra. More than this, it implies that the algebra of attributes is isomorphic with the algebra of *all* subsets of the universe whence, for example, it is atomistic, and satisfies the generalized distributive law.

156. The propositional calculus. A Boolean algebra of *propositions** can also be developed. To begin with, given propositions x, y, one can denote the propositions "x and y," "either x or y," and "not x," by $x \smile y$, $x \frown y$, and x' respectively. Under this notation, the identities of Boolean algebra relate logically equivalent statements. Thus (L2) "John sleeps and Henry walks" is true or false according as "Henry walks and John sleeps" is true or false. In summary,

THEOREM 8.3: *Propositions form a Boolean algebra.*

We can amplify the propositional calculus, by denoting the compound proposition "x implies y" ("if x, then y") by $x \rightarrow y$. Clearly $x \rightarrow y$ is true or false according as "y or not-x" is true or false; thus, following Whitehead and Russell, one can identify $x \rightarrow y$ with $x' \frown y$. Similarly, one can denote "x is equivalent to y" by $x \sim y$, and identify it with "x implies y and y implies x" i.e., with $(x \rightarrow y) \smile (y \rightarrow x)$.

Also, O denotes the proposition asserting nothing, and I the proposition which asserts everything. Incidentally, $x \sim y$ is the symmetric difference $x + y$, and $x \rightarrow y$ is $y - x$ (the part of y not included in x). Our notation is dual to the usual one.

One can draw a number of inferences from the above. For example, the compounding of n propositions with "and," "or," "not," "implies," and "is equivalent to," yields just 2^{2^n} propositions in all. Again (we shall omit the proof), one cannot construct "not x" or "x and y" from propositions of the form "u implies v." To have a binary operation from which all Boolean operations can be constructed, one needs something like Sheffer's stroke-operation $x|y = x' \smile y'$ meaning "neither x nor y."

One can also show that many compound propositions are "tautologies," that is, true merely in virtue of their logical structure. This amounts algebraically to saying that they are equal to O. The simplest of these is $x \frown x'$

* Synonyms for "proposition" are: "sentence," "statement," "theorem." The propositional calculus deals with compound sentences like: "John sleeps and Henry walks," "John sleeps or Henry walks," "John sleeps not." Incidentally, "x or y" is logically equivalent to "x unless y."

("x or not x"). It is a simple exercise in Boolean algebra to show that the following are also tautologies: (1) $I \to x$, (2) $x \to O$, (3) $x \to x$, (4) $x \sim x$, (5) $x \to (y \to x)$, (6) $(x \sim y) \sim (y \sim x)$, (7) $(x \sim O) \frown (x \sim I)$, (8) $(x \to a) \frown (a \to y)$, (9) $[(x \to y) \smile (y \to z)] \to (x \to z)$.

Tautology (7) asserts "any proposition is equivalent either to O or to I." Tautology (8) yields "of any two propositions, one implies the other." Tautology (1) states "a false proposition implies any proposition."

The three propositions about propositions (metamathematical propositions) just stated are disputed by some. Thus let x denote the proposition "Hitler rules Germany," and let y denote "Caesar conquered Gaul." To the ordinary man, to say that either of these implies the other, seems absurd; the two propositions are irrelevant. He would object even more strongly to the assertion that "Caesar did not conquer Gaul" implied "Caesar conquered Gaul"—although the hypothesis (being false) according to our theory implies every proposition, and the conclusion (being true) is implied by every proposition.*

But it seems to the author that the objection is invalid. If one is allowed to use the assumption "Caesar did not conquer Gaul," one can certainly prove that "Caesar conquered Gaul"; one can even prove it (by reference to historical records) without using this assumption.

One cannot prove it by pure logic without historical records, but the Whitehead-Russell propositional calculus is only intended for systems in which all propositions are demonstrably true or demonstrably false. Without historical records, the proposition could be neither proved nor disproved.

157. Critique of Boole's dual isomorphism. Boole's dual isomorphism affords a remarkable connection between the world of mind (attributes) and the dual world of matter (objects). But it leads to several perplexities.

In the first place, the notion of a "universe" is highly ambiguous. What one regards as "the set of all objects in the universe" varies with the period in which one lives, and also with one's state of knowledge. Thus it was changed by the telescope, by the discovery of the western hemisphere, and so on. This fact makes the construction of sums, products, and complements of subsets (e.g., "men," "double stars") of the universe quite hypothetical.†

Again, if one allows the "universe" to include attributes, one reaches a mathematical paradox. For Boole's correspondence would then make the universe include all its subsets as members. Its cardinal number $\aleph$ would then satisfy

* In this connection, the following anecdote is appropriate. Russell is reputed to have been challenged to prove that the (false) hypothesis $2 + 2 = 5$ proved, in particular, that he was the Pope. Russell reasoned as follows: "You admit $2 + 2 = 5$; but I can prove $2 + 2 = 4$. Taking 2 from both sides, we have $3 = 2$. Taking one more away, $2 = 1$. But you will admit I and the Pope are two. Therefore, I and the Pope are one, q.e.d."

† Indeed, it is remarkable that so many attributes have extended meanings when the universe is widened. Thus the attribute of being "human" (as distinct from being a member of the class of known men) extended naturally from the eastern to the western hemisphere. Actually, it is questionable whether one can construct the complement of an infinite set, in general.

$2^{\aleph} \leqq \aleph$, and this can be disproved by generalizing Cantor's diagonal process.*

It is also questionable whether the correspondence $x \rightarrow \hat{x}$ is isomorphic or homomorphic. Thus let x and y be the attributes of being (living) dinosaurs and centaurs respectively. Surely $\hat{x} = \hat{y} = O$, yet one would hardly identify the two concepts!†

Other considerations should be mentioned. For example, can every attribute be decomposed into "categorical" attributes, possessed by just one object? or into attributes possessed by all objects except one? Again, are there no attributes (of degree) varying continuously with a real parameter? If one swallows Boole's theory, one is forced to accept these conclusions. One is also confronted with a system on which no countably additive probability function can be erected.‡

In view of these considerations, it is imperative to have a formulation of the algebra of logic which is *independent* of set theory (i.e., axiomatic). An adequate axiomatic formulation is furnished by the preceding chapters of this book.

It is also desirable to have models for the algebra of attributes differing from Boole's model of the algebra of all subsets of a supposed "universe." The following sections will be devoted to exhibiting such alternative models.

158. A model from classical mechanics. Two models are suggested by mathematical physics; the first by the n-body problem, as we shall now see.

The "state" of a system Σ of n bodies§ at any time t_0, can be expressed by $6n$ numbers: each body has three coordinates of position and three of velocity. Moreover this description is complete, in the sense that the state of Σ at any later (or earlier!) time is determined by these numbers and the laws of mechanics.

In summary, the "state" (or phase) of Σ can be represented by a point in $6n$-dimensional Cartesian space, and this space is called the *phase-space* of Σ.

When Σ is under discussion, its phase-space is clearly a sort of "universe of discourse" I. Each attribute‖ x of Σ determines a set x in I: the set of all "states" in which Σ has the given attribute.

But no consequential theory is known which permits *every* subset of I to correspond to an attribute. Physically, the accuracy of measurements is limited,

* This paradox can be avoided by using de Morgan's concept of a limited "universe of discourse." In this, attributes are applied only (say) to men, to animals, to colors, or to members of some other restricted class. Indeed, most adjectives (viz., angry, blonde) apply only to a limited domain of nouns.

† If one decides that the correspondence is homomorphic, one might with propriety call attributes "objectively equivalent" when (like equilateral and equiangular triangles) they are possessed by identical classes of objects.

‡ The atomic and discontinuous nature of the algebra of all subsets of any aggregate is obvious. For the last paradox, cf. S. Ulam, *Zur Masstheorie in der allgemeinen Mengenlehre*, Fund. Math., 16 (1930).

§ These bodies may either represent the planets and their satellites (astronomy), or the molecules of a gas (kinetic theory of gases). Phase-spaces are also used in thermodynamics.

‖ As examples of attributes, one may cite having an eclipse from astronomy—or being below atmospheric pressure for a confined gas.

so that (for example) one could never know by experiment whether the kinetic energy of Σ was a rational quantity, measured in ergs. Even mathematically, no theory could assign to *every* attribute a probability (in statistical mechanics), and the ergodic theorem is only proved for measurable subsets.

Without going into too lengthy a discussion here, we may assert that dictates of mathematical necessity and of physical plausibility* strongly suggest letting "attributes" correspond to measurable subsets, ignoring sets of measure zero.

The resulting model for the algebra of logic is of course the M/N of §122. It is (a) a complete Boolean algebra, which (b) fails to satisfy the unrestricted distributive law, (c) is without "categorical" attributes, and (d) admits a non-trivial, countably additive probability function. It is, incidentally, the model on which Kolmogoroff bases his theory of probability.

159. A model from quantum mechanics. An analogous situation prevails in quantum mechanics. Under the present mathematical theory,† the "state" of a system Σ is represented by a point in the space (L_2) (Hilbert space), which thus acts as a "phase-space" I.

Moreover if x is any observable attribute, the set $\hat{x}$ of states in which Σ is certain to be observed to have the attribute x forms a *closed subspace* of I. The statement "possession of attribute x implies possession of attribute y" is equivalent to the set-theoretical relation $\hat{x} \leqq \hat{y}$, and the negation of x corresponds to the *orthogonal complement* of $\hat{x}$. (The proof of these statements involves the usual assumption that all the operators of quantum mechanics are Hermitian linear operators.)

Now suppose one admits the following

Hypothesis: The linear sum of any two subspaces which correspond to observable attributes itself corresponds to an observable attribute.

It will follow that any linear sum $X + Y$, orthogonal complement X', or intersection $X \frown Y = (X' + Y')'$ of subspaces corresponds to an observable attribute.

Thus we get an algebra of observable attributes which is an *orthocomplemented modular lattice.* Moreover under the present theory, the distributive law is satisfied by simultaneously observable attributes, but not others.‡ Incidentally,

* Von Neumann has remarked that since a set is measurable if and only if it has density 0 or 1 almost everywhere, we select precisely those attributes upon whose truth or falsehood we can pronounce with arbitrary nearness to certainty, by making sufficiently accurate measurements. Also, all known attributes (viz., of having temperature, pressure, etc., within fixed limits) correspond to Borel sets.

† Cf. J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*, Berlin, 1931, or P. A. M. Dirac, *Wave Mechanics*, Oxford University Press, 1930. The ideas of the present section are due to von Neumann and the author, *The logic of quantum mechanics*, Annals of Math., 37 (1936), 823–43.

‡ This is a well-known and deep phenomenon. Quantities are simultaneously observable if and only if their operators permute, which is the case if and only if their spectral resolution yields a single chain (distributive lattice) of characteristic subspaces.

This may be related to the fact that for simultaneously observable attributes, "x and y or z" is obviously equivalent to "(x and y) or (x and z)."

the complemented lattice of *all* closed subspaces is a complete lattice, but is neither modular nor a sublattice of the lattice of all subspaces. Hence our Hypothesis does not apply to it.

These conclusions suggest that for certain atomic systems, the algebra of attributes may be isomorphic with von Neumann's continuous geometry over the real field.

This would yield a beautiful analogue to the model M/N of §158. It is (a) a complete, complemented, modular lattice, which (b) satisfies the distributive law only for simultaneously observable attributes, (c) is without "categorical" attributes, and (d) admits a non-trivial, countably additive dimension function.

Incidentally, a recent theory of psychological tests has been proposed, which remotely suggests—at least to the author—a similar model for mental traits.†

160. Intuitionist logic and Lewis' strict implication. Brouwer and his "intuitionist" school have proposed a quite different alteration of Boole's system. They reject the "law of the excluded middle," according to which if "p" is any proposition, either "p" or "not p" must be true. In the same spirit, they reject proofs by contradiction, for the disproof (or "reductio ad absurdum") of "not p" need not imply the truth of "p" if there is a third possibility.

And indeed, there is some evidence in favor of Brouwer's point of view. The existence of "undecidable" propositions (for which neither "p" nor "not p" is demonstrable) seems to have been established by Skolem and Gödel.‡

This denial that "not not p" implies "p" (the reverse implication is however admitted) is the most conspicuous feature of intuitionist logic, and of C. I. Lewis' system of "strict implication" as well.§ Interpreted formally, it is the denial of the identity $(x')' = x$, and substitution for it of the inequality $(x')' \leqq x$. In the next sections, we shall develop the theory from an algebraic standpoint.

161. Algebraic theory. A somewhat unorthodox basis for the "intuitionist" algebra of propositions, but one which is a tremendous simplication from the lattice-theoretic point of view, is contained in

DEFINITION 8.1: *By a "Brouwerian logic," will be meant a lattice with O and I, in which an operation $x \rightarrow y$ is defined, such that*

B1: $(x \rightarrow y) = O$ *if and only if* $x \geqq y$,
B2: $x \rightarrow (y \rightarrow z) = (x \cup y) \rightarrow z$.

The element $x \rightarrow I$ will be denoted x^*, and the relation $x = O$ by $\vdash x$.

† L. L. Thurstone, *Vectors of Mind*, Chicago University Press, 1935. Among other things, the relation of linear dependence among mental attributes or "traits" is defined.

‡ K. Gödel, *Über unentscheidbare Sätze* ···, Monats. f. Math. u. Phys., 38 (1931), 173–98. Such a conclusion depends of course on prescribing all admissible methods of "proof." For instance, Carnap has stated plausible methods of proof excluded by Gödel. Hence such a conclusion should be viewed with deep skepticism.

§ Cf. A. Heyting, *Mathematische Grundlagenforschung, Intuitionismus, Beweistheorie*, Berlin, 1935, and C. I. Lewis and C. H. Langford, *Symbolic Logic*, New York, 1932. Also Gr. C. Moisil, *Recherches sur le syllogisme*, Annales Sci. de Jassy, 25 (1939), 341–84.

LEMMA 1: *$x \to y$ is the least t satisfying $t \smile x \geqq y$.*

Proof: By B1, $t \geqq (x \to y)$ is equivalent to $\vdash t \to (x \to y)$; by B2, this is $\vdash (t \smile x) \to y$; and by B1 again, this is equivalent to $t \smile x \geqq y$.

LEMMA 2: *A Brouwerian logic is a (distributive) lattice in which the meet $x \to y$ of the t satisfying $t \smile x \geqq y$ exists, and itself satisfies $(x \to y) \smile x \geqq y$. Conversely, any lattice in which this condition holds is a base for a Brouwerian logic.*

Proof: The first statement, apart from distributivity, follows from Lemma 1. As regards this $y \smile x \geqq (x \smile y) \frown (x \smile z)$ and $z \smile x \geqq (x \smile y) \frown (x \smile z)$; hence by what we have just said

$$(y \frown z) \smile x \geqq (x \smile y) \frown (x \smile z).$$

Now use Theorem 5.1.

Conversely, in such a lattice L with $x \to y$ defined as in Lemma 1, (B1) $(x \to y) = O$ if and only if $x \smile O \geqq y$, and (B2) since $t \smile (x \smile y) \geqq z$ if and only if $(t \smile x) \smile y \geqq z$—i.e., if and only if $t \smile x \geqq y \to z$— we have $t \geqq (x \smile y) \to z$ if and only if $t \geqq x \to (y \to z)$.

Combining Lemmas 1–2 one obtains readily the

THEOREM 8.4: *A complete lattice is a base for a Brouwerian logic if and only if the following distributive law holds: $a \smile \wedge x_\alpha = \wedge(a \smile x_\alpha)$. In this case, $x \to y$ is necessarily the least t satisfying $t \smile x \geqq y$.*

One corollary of this is the fact that any finite distributive lattice is a base for a Brouwerian logic. Another corollary is the fact that the system of closed subsets of any topological space (technically, T_0-space) has this property. This contains the essence of recent results of Tarski and Tang.†

LEMMA 3: *Any Brouwerian logic satisfies Heyting's axioms for "intuitionist logic," and also the axioms of C. I. Lewis for "strict implication."*‡

Proof: In our notation, Heyting's assumptions are $\vdash a \to (a \smile a)$, $\vdash a \smile b \to b \smile a$, $\vdash (a \to b) \to (a \smile c \to b \smile c)$, $\vdash [(a \to b) \smile (b \to c)] \to (a \to c)$, $\vdash b \to (a \to b)$, $\vdash [a \smile (a \to b)] \to b$, $\vdash a \to (a \frown b)$, $\vdash a \frown b \to b \frown a$, $\vdash [(a \to c) \smile (b \to c)] \to (a \frown b \to c)$, $\vdash a^* \to (a \to b)$, $\vdash [(a \to b) \smile (a \to b^*)] \to a^*$. Using the remark that by B1, $\vdash f \to g$ is equivalent to $f \geqq g$, it is not hard to verify the different laws.

A similar discussion applies to strict implication; in fact, here there are many fewer assumptions to verify. Lewis' $\Diamond\, p$ should be interpreted as $p < I$.

† Cf. M. H. Stone [5]; A. Tarski, *Der Aussagenkalkul und die Topologie*, Fund. Math., 31 (1938), 103–34; Tsao-Chen Tang, *Algebraic postulates and a geometric interpretation for the Lewis calculus of strict implication*, Bull. Am. Math. Soc., 44 (1938), p. 737.

‡ Cf. A. Heyting, *Die formalen Regeln der intuitionistischen Logik*, S.-B. preuss. Akad. Wiss. (1930), 42–56; Lewis and Langford, op. cit., p. 493. Only Lewis' definition of $x \frown y$ as $(x^* \smile y^*)^*$ is not satisfied. But as this makes $x \frown x = (x^*)^*$, perhaps the fault is with his definition!

162. Brouwerian vs. classical logic. One can now prove without trouble the following laws of "intuitionist" logic, which are weakened laws of classical logic,†

L81: $(x^*)^* \geqq x$. L82: *If* $x \geqq y$, *then* $y^* \geqq x^*$.
L83: $(x \frown y)^* = x^* \smile y^*$. L84: $(x \smile y)^* \leqq x^* \frown y^*$.
L85: $((x^*)^*)^* = x^*$.

Proof of L81: By Lemma 1, $x \smile (x \to I) \geqq I$; hence $((x \to I) \to I) \geqq x$—that is, $(x^*)^* \geqq x$. Proof of L82: By hypothesis, $x \smile y^* \geqq y \smile y^* = I$; hence $y^* \geqq (x \to I)$ by Lemma 1. Proof of L83: $t \geqq (x \frown y)^*$ is by Lemma 1 equivalent to $(x \frown y) \smile t = I$. But $(x \frown y) \smile t = (x \smile t) \frown (y \smile t)$; hence $t \geqq (x \frown y)^*$ is equivalent to $x \smile t = y \smile t = I$—i.e., to $t \geqq x^*$ and $t \geqq y^*$, or, to $t \geqq x^* \smile y^*$. L84 follows from L82, and L85 from L81 and L82.

Only a very slight strengthening of the laws of Brouwerian logic is needed to obtain the classical (Boole-Whitehead) logic of §156; either $(x^*)^* = x$ or $\vdash x \frown x^*$ will do it. For if $(x^*)^* = x$, then by L82 the correspondence $x \to x^*$ is an *involution* (dual automorphism of period two), whence $x \smile x^* = I$ implies $x^* \frown x = O$, our second assumption. But this together with $x \smile x^* = I$ implies that our distributive lattice is complemented—i.e., a Boolean algebra. And on a Boolean algebra by Theorem 8.4 $x \to y$ must have the meaning $x^* \frown y$.

Brouwerian logic contains classical logic in it; the propositions which are their own double negatives form a classical propositional calculus.‡ The correspondence $x \to (x^*)^*$ maps any Brouwerian logic in an order-preserving way on this subsystem. This correspondence is lattice-endomorphic by L83 if and only if $(x \smile y)^* = x^* \frown y^*$—a condition holding in chains, but not in the B_1 of Fig. 13.

One more distinction may be drawn between classical and Brouwerian logic. In the former, the operations $\smile$, $\to$ can be expressed as functions of the operations $\frown$, $'$ thus: $(x \smile y) = (x' \frown y')'$ and $x \to y = x' \frown y$. Whereas in Brouwerian logic, this is impossible. Thus in B_2 of Fig. 13, the set consisting of $O, x \frown y, x, y, I$ is closed under the operations $\frown$, $*$, $\to$, but not under $\smile$; the set consisting of $O, x \frown y, x, x \smile y, I$ is similarly closed under $\frown$, $*$, $\smile$, but not under $\to$. Thus neither $\smile$ nor $\to$ can be expressed even as functions of $\frown$, $'$ and the other.

163. Modal logic. In §160 the assumption that some propositions were neither true nor false, but lay between these two extremes, was used to justify Brouwerian logic and C. I. Lewis' strict implication. Theories of logic which start from this assumption, constitute what is usually called "modal logic."

In modal logic, the categories into which propositions fall are called "modes",

† Note especially the non-duality of L83, L84, which correspond to laws (7), (8), (11) of Heyting's Ergebnisse monograph, p. 15. A similar model is furnished by the subspaces of Hilbert space, letting x^* denote the orthocomplement of x. Cf. M. H. Stone, *Linear Transformations in Hilbert Space and their Applications to Analysis*, New York, 1932, esp. p. 20; also §159.

‡ V. Glivenko, *Sur quelques points de la logique de Brouwer*, Bull. Acad. Sci. Belg. (1929), 183–88; more especially K. Gödel, Ergebnisse eines Kolloquiums, Vienna, 4 (1933), 35–40.

or "truth-values"; but there are many schools of thought regarding what these categories are.

Thus in traditional Aristotelian logic, there are four modes: necessary, contingent, possible, impossible. In modern logic, there seem to be three: true,

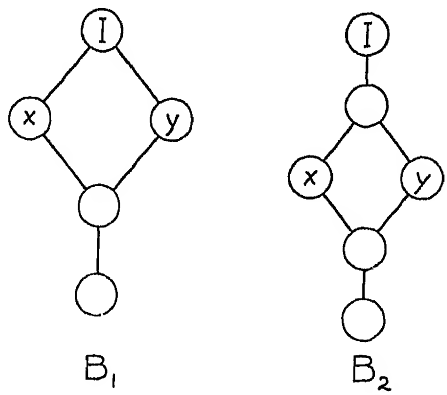

FIG. 13

undecidable, and false. In the classical theory of probability, the scale of "truth-values" runs through the real numbers between zero and one. A propositional calculus which can also be viewed as having these modes has been proposed by Lukasiewicz and Tarski.*

In each of these theories, the "truth-values" or "modes" are *simply* ordered by the degree of truth which they ascribe to propositions; thus they form a scale. But there is no reason why modes should not constitute a *partially* ordered set; indeed, Keynes has advanced precisely this view, as coming closer to reality.†

The author knows of no study of propositional calculi based on (for example) truth-values forming very simple non-distributive lattices. In attempting to construct these himself, he has been troubled by the fact that except in the two-valued case, the correspondence between propositions and truth-values is not homomorphic: thus the truth-values of p and q do not determine the truth-value of $p \to q$.

For this and other reasons, attention will be confined to the classical theory of probabilities, which is by far the most important case.‡

* Cf. O. Frink, *New algebras of logic*, Am. Math. Monthly, 45 (1938), p. 212.

† *A Treatise on Probabilities*, London, 1929, p. 39. This view is shared by B. O. Koopman, *The axioms and algebra of intuitive probability*, Annals of Math., 41 (1940), 271–92. A detailed discussion of several "truth-value systems" is given by Lewis and Langford, op. cit., Chap. VII; cf. also the end of Gr. C. Moisil's article, *Recherches sur l'algèbre de la logique*, Annales Sci. de l'Univ. de Jassy, 22 (1936), 1–118.

‡ The author's opinion that probability may properly be regarded as a kind of modal logic was shared by J. Venn, who said "the modals are the nearest counterpart to modern probability which was afforded by the old systems of logic" (*The Logic of Chance*, London, 1888).

CHAPTER IX

APPLICATIONS TO PROBABILITY

164. Definition of probability functional. The general theory of probability can be based on the following

DEFINITION 9.1: *By a "distribution" or "probability functional" on a Boolean algebra A, is meant a positive, modular functional $p[x]$ which satisfies $p[O] = 0$, $p[I] = 1$.*

Remark: The theory developed below holds equally well for complemented modular lattices, formally. Indeed, since (Theorem 3.11) any modular functional converts a lattice into a (metric) modular lattice, we could even allow A to be any complemented lattice. But no applications of this generalization are known. Quantum mechanics (§159) seems the most likely source for such applications.

Definition 9.1 yields immediately the following two well-known laws: (1) $p[x'] = 1 - p[x]$, and (2) if $x_i \frown x_j = O$ $[i \neq j]$, then $p[x_1 \smile \cdots \smile x_n] = p[x_1] + \cdots + p[x_n]$.

It may also be remarked that Tornier's postulates for the theory of probability are equivalent to the conditions of Definition 9.1.*

165. Examples. One can illustrate Definition 9.1 very easily by well-known examples.

Example 1: Let an experiment E have possible eventualities $h_1, \cdots, h_m$. Let A be the Boolean algebra of all subsets x of the class of h_i. Finally, let $p_n[x]$ denote the proportion of the first n trials of E yielding eventualities in the set x. Then Definition 9.1 is satisfied.

Thus E might be the drawing of a poker hand, and the h_i the 2,598,960 possible poker hands. Or again, E might be the issuance of a life insurance policy, and h_k the eventuality that the insured died after paying k premiums.

Example 2: Imagine an experiment E performed infinitely often in succession. Let h_i, A, and $p_n[x]$ be defined as in Example 1. Assume that as $n \to \infty$, every $\{p_n[x]\}$ converges to a limit $p_\infty[x]$. Then $p_\infty[x]$ satisfies Definition 9.1.

This hypothetical situation is the foundation of the theory of probability according to the school of von Mises.

Example 3: Let a disc of circumference unity be spun infinitely often. Let A consist of the Jordan subsets x of the circumference. Let $p_n[x]$ be the proportion of the first n spins after which a fixed pointer indicates a point in x. Then the $p_\infty[x]$ of Example 2 satisfies Definition 9.1, provided it exists.

Considerations of symmetry suggest that it will be the measure of x.

* E. Tornier, *Wahrscheinlichkeitsrechnung und allgemeine Integrationstheorie*, Leipzig, 1936.

Example 4: Let A be the Boolean algebra M/N of §121, and let $p[x]$ be the measure of x. Then $p(x)$ satisfies Definition 9.1—and may be thought of as the probability that a point, thrown into I at random, will land* in x.

Remark: Any probability functional $p[x]$ on a finite Boolean algebra A, corresponds to a real example. For by §90, A can be realized isomorphically by sets on the circumference of a circular disc, and in such a way that the set corresponding to any x is a sum of sectors of total length $p[x]$. Now set a fixed pointer next to the disc. Then the probability that after spinning the disc will come to rest so that the pointer indicates a point of x is (under the orthodox theory) precisely $p[x]$.

166. The algebra of probability. We now come to the algebra of probability functions on a fixed Boolean algebra A. Here we run into the curious anomaly that neither the join $p \smile q$, meet $p \frown q$, nor sum $p + q$ of distinct probability functions, is itself a probability function!

In order to get an algebra for probability functions, one must enlarge the class of these to the set $L(A)$ of all *differences* $\lambda p - \mu q$ of *multiples* of probability functions. Then one can show

THEOREM 9.1: *The algebra $L(A)$ of probability consists of the bounded modular functionals on A satisfying $f[O] = 0$. It is a "space (AL)":† that is, a Banach lattice in which $f > 0$, $g > 0$ imply $|f + g| = |f| + |g|$.*

The proof of this theorem can conveniently be made by several stages, each involving an easily proved lemma.

LEMMA 1: *Let $f[x]$ be a modular functional on a complemented modular lattice A. Then the variation of $f[x]$ on any chain $a = x_0 < x_1 < \cdots < x_n = b$ is equal to that on some chain $a \leqq y \leqq b$ of length two.*

Proof: Let y_i $[i = 1, \cdots, n]$ be any relative complement of x_{i-1} in x_i. Let y be the join of a with the y_i such that $f[x_i] \geqq f[x_{i-1}]$. Then $f[y] - f[a]$ and $f[b] - f[y]$ are by M1 and induction the positive and negative parts of the variation of $f[x]$ on the chain, respectively.

COROLLARY: *In Lemma 1, f is of finite variation if and only if it is bounded. If $f[O] = 0$, then in addition $f^+[I] = \sup f[x]$, $f^-[I] = \inf f[x]$, and the total variation of f is $\sup f[x] - \inf f[x]$. Finally, $f^+[a]$ and $f^-[a]$ are respectively the $\sup$ and $\inf$ of $f[x]$ on $0 \leqq x \leqq a$.*

LEMMA 2: *$L(A)$ includes only bounded modular functionals f on A satisfying $f[O] = 0$.*

* Such examples in so-called geometrical probabilities establish a direct isomorphism between the theory of measure and that of probability.

† The author coined the phrase "space (L)" ([8]) because the spaces (L) and (l) of functional analysis were examples satisfying the definition. The phrase "measure space" might be equally appropriate. Other examples include the space (V) of functions of bounded variation, with total variation as norm, and the space (AP) of almost periodic functions, with norm equal to the mean of $|f|$.

Proof: The conditions of boundedness, modularity, and vanishing at 0 are separately trivial.

LEMMA 3: *The functionals described in Lemma 2 form a linear space, and precisely constitute* $L(A)$.

Proof: That they form a linear space is trivial (the sum of bounded, of modular functionals is bounded, modular, etc.). Moreover any such functional has by §55 a Jordan decomposition into its positive and negative variations,* thus: $f = f^+ + f^-$. Finally, f^+ and f^- are multiples of *distributions* by $f^+[I]$ and $f^-[I]$; hence every functional described in Lemma 2 is in $L(A)$.

LEMMA 4: $L(A)$ *is a vector lattice.*

Proof: Clearly if $h \geqq 0$ and $h \geqq f$, then $h \geqq f^+$, where f^+ denotes the positive variation of f. Conversely, $f^+ \geqq 0$ and $f^+ \geqq f$. Hence the positive variation of f is $f \smile O$. Now apply Theorem 7.2.

LEMMA 5: $L(A)$ *is a Banach space if the total variation* $|f| = f^+[I] - f^-[I]$ *is used as norm.*

Proof: The identities $|f + g| \leqq |f| + |g|$ and $|\lambda f| = |\lambda| \cdot |f|$ are trivial in view of the corollary of Lemma 1. Again, if $|f_m - f_n| \to 0$ as $m, n \to \infty$, then for all x, $|f_m[x] - f_n[x]| \to 0$ as $m, n \to \infty$, being bounded by $|f_m - f_n|$. Hence for each x, the $f_n[x]$ converge to some $g[x]$. The modularity and boundedness of g, and the relation $g[O] = 0$, follow by continuity; so does $|f_n - g| \to 0$. Hence $L(A)$ is a Banach space.

LEMMA 6: *If* $f > 0$ *and* $g > 0$, *then* $|f + g| = |f| + |g|$. *Also, for any* f, $|f| = \|f\|$; *i.e.,* $L(A)$ *is a Banach lattice.*

Proof: If $h > 0$, by Lemma 1 $|h| = h[I]$; in the light of this, the first assertion is trivial. Using this and $|f| = f^+ + (-f)^+$, we get

$$\|f\| = \sup f[x] + \sup (-f[x]) = \sup f[x] - \inf f[x].$$

But the last difference is $|f|$, by Lemma 1.

167. The subset of distributions. It is easy to describe the location of the set D of distributions (alias probability functions) in the space $L(A)$.

THEOREM 9.2: *The set* D *of distributions consists of the positive elements of* $L(A)$ *of norm one. It is therefore metrically closed, convex, and of diameter at most two.*

Proof: By definition, D consists of the positive elements satisfying $f[I] = 1$. But if $f > 0$, then $|f|$ is $f[I]$, proving the first statement. Again, both the set of $f \geqq 0$ (Theorem 7.20) and the "unit sphere" of f of norm unity are metrically closed; hence so is their intersection D. While if $p > 0$, $q > 0$, $\lambda > 0$, $\mu > 0$,

* $f^+[x]$ denoting of course the positive variation of f on the interval $0 \leqq t \leqq x$, and $f^-[x]$ being defined dually.

$\lambda + \mu = 1$, then $\lambda p + \mu q > 0$; and if besides $|p| = |q| = 1$, then $|\lambda p + \mu q| = \lambda|p| + \mu|q| = 1$. Finally, if $p, q \in D$, then $|p - q| \leqq |p| + |-q| = 2$.

Remark 1: The distance function $|p - q|$ is the "stochastic distance" of Mazurkiewicz;* convergence with respect to this metric is *not* equivalent to the traditional notion of "convergence in probability," but implies it.

Remark 2: The functional $m[f] = f[I]$ is positive and modular on $L(A)$. The metric definable as in §51 from this functional gives one $|f - g|$. More generally, every $x \in A$ defines an element $m_x[f] = f[x]$ of the conjugate space of $L(A)$. Thus the conjugate space of $L(A)$ contains A as a subset.

Remark 3: Except in trivial cases, D contains elements satisfying $p \frown q = O$, for which $|p - q| = 2$. That is, the diameter of D is *exactly* two, except in trivial cases.

168. Continuous distributions. Now suppose A is σ-complete; what is the nature of the subset of $L(A)$ consisting of *continuous* distributions?†

LEMMA: *Each of the following conditions is equivalent to continuity*: (1) $x_n \downarrow 0$ *implies* $f[x_n] \to 0$, (2) *countable additivity—i.e.,* $V_{i=1}^{\infty} x_i = x$ *and* $x_i \frown x_j = O$ $[i \neq j]$ *imply* $f[x] = \sum_{i=1}^{\infty} f[x_i]$.

Proof: Since $y_n \to y$ if and only if $x_n \downarrow 0$ exists satisfying $|y_n - y| \leqq x_n$ (cf. Definition 2.3; $|y_n - y|$ denotes the symmetric difference between y_n and y), condition (1) is equivalent to continuity. Again, if $x_i \frown x_j = O$ $[i \neq j]$, then $f[V_{i=1}^{n} x_i] = \sum_{i=1}^{n} f[x_i]$. Hence if f is continuous, so that $f[V_{i=1}^{n} x_i] \to f[V_{i=1}^{\infty} x_i]$, it is countably additive; that is, $f[V_{i=1}^{\infty} x_i] = \sum_{i=1}^{\infty} f[x_i]$. Conversely, since $x_n \downarrow O$ implies $x_1 = V_{n=1}^{\infty}(x'_{n+1} - x'_n)$, where $(x'_{i+1} - x'_i) \frown (x'_{j+1} - x'_j) = O$ $[i \neq j]$, countable additivity implies $f[x_1] = \lim_{n\to\infty}(f[x_1] - f[x_n])$ —i.e., $f[x_n] \to 0$. Thus countable additivity implies condition (1), and so continuity.

THEOREM 9.3: *If A is σ-complete, the set $L_c(A)$ of* continuous *members of $L(A)$ is a metrically closed, normal subspace of $L(A)$.*

Proof: Since $|f[x_n]| \leqq |f_m - f| + f_m[x_n]$, if every f_m is continuous and $|f_m - f| \to 0$, then $f[x_n] \to 0$; that is, $L_c(A)$ is metrically closed. That $L_c(A)$ is a subspace of $L(A)$ is trivial; it remains to show that it is normal. But indeed, suppose $f \in L_c(A)$ and $0 < g \leqq |f|$. Then $x_n \downarrow 0$ and $0 \leqq y_n \leqq x_n$ imply $y_n \to 0$ and so $f[y_n] \to 0$; hence $\sup_{0 \leqq y_n \leqq x_n} |f[y_n]| \to 0$. But by hypothesis and Lemma 1 of §167, $|g[x_n]| \leqq |f|[x_n] \leqq 2 \sup_{0 \leqq y_n \leqq x_n} |f[y_n]|$; therefore $g(x_n) \to 0$ and g is continuous (by the Lemma), completing the proof.

COROLLARY 1: *The space $L_c(A)$ of bounded, continuous, modular functionals on A is a space (AL).*

* *Über die Grundlagen der Wahrscheinlichkeitsrechnung*, Monats. f. Math. u. Phys., 41 (1934), 343-53. The metric for "convergence in probability" does not even yield a Banach space.

† It will be recalled that a functional $f[x]$ on A is called "continuous" if and only if $x_n \to x$ in the (o)-topology implies $f[x_n] \to f[x]$. Also, if A is finite, every member of $L(A)$ is trivially continuous.

COROLLARY 2: *The continuous distributions on A form a metrically closed, convex subset D_c of $L_c(A)$.*

It should be remarked that Kolmogoroff identifies the theory of probability with the general theory of continuous (i.e., countably additive by our Lemma) distributions on σ-complete Boolean algebras.†

169. Spaces (L) as phase-spaces. There are many situations in which one's knowledge of a system Σ —that is, one's ability to predict the properties of observations on Σ —is limited to *probable* assertions.

Example 1: Two dice are to be picked up and thrown. We know that the number of pips showing will exceed four, with probability five-sixths.

Example 2: A new pack of cards, arranged by suits and within each suit in order A23 $\cdots$ JQK, is shuffled twice. We know that the two and three of hearts will "probably" be very near together.

Example 3: A tiny particle suspended in a liquid undergoes "Brownian movement." We know that its position will "probably" alter very little in ten seconds, under the random bombardment of molecules.

Example 4: A Geiger counter records cosmic rays. If an average of ten per minute are recorded, we know that there is a probability $1/(n!)e$ that exactly n will be counted in a given six-second interval.

Example 5: In quantum theory, a particle with given ψ-function has a probability

$$p[R] = \int_R \psi\psi^* dV$$

of being observed in a region R.

Example 6: Even in classical (deterministic) mechanics, limitations on the precision of instruments make predictions subject to error. This is well-known, and is assumed to take the form of a so-called Gaussian distribution.

Thus in every case one's state of knowledge seems to be best described by a probability functional $p[x]$, expressing the probability that a given observation on Σ will have property x. But we have seen that whether or not continuity (continuous distributions on continuous lattices) is required, these yield a space (AL). This leads one to consider spaces (AL) as "phase-spaces" (cf. §158), points in which represent conceivable states of knowledge.

170. Transition operators. Now consider the dependence of one's state of knowledge q_0 about a future observation on one's knowledge p_0 about the present. A *law* describing this dependence, clearly amounts mathematically to a *transformation* T: $p \rightarrow pT = q$ of our space into itself—i.e., to an *operator* on our phase-space (AL).

The same situation arises in both classical and quantum mechanics: in both, one has a "phase-space" of "states," and the dependence of the future on the

† A. Kolmogoroff, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Berlin, 1933, Ergebnisse series.

We note that the space (L) is $L_c(M/N)$.

present is expressed by operators on this space. It brings us to the question: what are these operators like, in the case of phase-spaces (AL)?

This question has already been answered in special cases. In all cases where a phase-space (AL) is used,† the operator is a "transition operator," in the sense of

DEFINITION 9.2: *By a "transition operator" on a space (AL), is meant an additive operator which carries distributions (i.e., positive elements of norm one) into distributions.*

Example 1: Let $L(B^n)$ be the n-dimensional space of sequences $f = [f_1, \cdots, f_n]$. Then the transition operators on $L(B^n)$ are the matrices $T = \| t_{ij} \|$ such that (1) $t_{ij} \geqq 0$ for all i, j, (2) $t_{i1} + \cdots + t_{in} = 1$ for all i—where $fT = [\sum_i f_i t_{i1}, \cdots, \sum_i f_i t_{in}]$.

In other words, in the finite-dimensional case, transition operators are what are usually called "matrices of transition probabilities." These are well-known;‡ it is precisely their behavior which is discussed in the theory of finite "dependent probabilities."§

Example 2: Let $L_e(M/N)$ be the ordinary space (L); let $K(x, y)$ be continuous and non-negative on $0 \leqq x, y \leqq 1$. Further, let $\int K(x, y)dx = 1$ for all y. Then the equation $g(x) = \int K(x, y)f(y)dy$ defines a transition operator $f \to g$.

These transition operators are those used by Kolmogoroff in his theory of "stochastic processes." Their relation to the theories of diffusion (Brownian movement), and of heat conduction, is discussed in Kolmogoroff's papers.‖

In this connection we may remark a direct connection between Definition 9.2 and the laws of thermodynamics, quite apart from Fourier's differential equations for heat. Let the thermal energy of each region R of an insulated solid I be $h(R)$; let the thermal energy five minutes later be $h^*(R)$. Then conservation of energy makes $h(I) = 1$ imply $h^*(I) = 1$, and the second law of thermodynamics implies that everywhere positive temperature (relative to any zero) is preserved.

Example 3: In Example 6 of §169 (classical dynamics), the volume-preserving

† This excludes quantum mechanics (Example 5 above). Here the emphasis on spectral theory has led to using Hilbert space as the phase-space. In fact, one has to consider ψ instead of the distribution p, to get the laws.

‡ Their relation to probability is easy to explain. Let a system be capable of assuming n states $1, \cdots, n$. (These might be the 52! arrangements of a pack of cards, or the eigen-states of an atom.) Let t_{ij} denote the probability that the state i give place to state j after some episode (e.g., one shuffle, or a lapse of 1/1000 second). Then $T = \| t_{ij} \|$ is a "matrix of transition probabilities."

§ Cf. M. Fréchet, *Méthodes des Fonctions Arbitraires, Théorie des Événements en Chaîne* ···, Paris, 1938, where further references to the immense literature on this subject may be found.

‖ A. Kolmogoroff, *Die analytische Methoden der Wahrscheinlichkeitsrechnung*, Math. Ann., 104 (1931), 415–58, and *Zur Theorie der stetigen zufälligen Prozesse*, ibid., 108 (1933), 149–60. His integral kernels are obvious analytical analogues to the matrices of Example 1; but his definition is less general than that of a "transition operator."

flow of ordinary phase-space Σ yielded by the laws of dynamics,* generates an automorphism of the space (L) of Lebesgue-integrable functions on Σ.

In summary, our definition of a "transition operator" includes all of the operators used in the theory of dependent probabilities, avoids the anomaly of requiring separate definitions for the finite and analytical cases, and besides includes the "deterministic" case of classical mechanics.†

171. Geometrical aspects. Our definition also makes possible some illuminating geometrical arguments. Thus

THEOREM 9.4: *Transition operators are either* isometries *or* contractions:‡ $|fT - gT| \leqq |f - g|$.

Proof: Since $fT - gT = (f - g)T$, we need only show that $|hT| \leqq |h|$. But $|h^+T| = |h^+|$ and $|h^-T| = |h^-|$, since T carries positive (and so negative) elements of norm one into similar elements. Hence

$$|hT| = |h^+T + h^-T| \leqq |h^+| + |h^-| = |h|.$$

A further study enables one to characterize various important special cases. Thus the "tychistic" case of "independent probabilities" (where pT is independent of p, as in Example 1 of §169) is the case where $D(A)$ is contracted to a point, and the phase-space $L(A)$ projected onto the axis through this. Again, in the antipodal deterministic case of classical mechanics, we have an *isometry* of the space (L): distances are preserved.

It is the intermediate "stochastic" case which is typical of the theory of dependent probabilities: here the set D of distributions is contracted somewhat, but not to a point. The case usually characterized laboriously by matrix theory as the one in which Markoff's Theorem (§174) holds is simply the case of *uniform contraction* (all distances on D being shrunk in some ratio $1:1 - \epsilon$ $[\epsilon > 0]$ at least).

172. Stable distributions. Finally, a distribution p is "stable" in the usual (natural) sense, if and only if it is a *fix-point* of T—that is, if and only if $pT = p$.

THEOREM 9.5: *The set of points left fixed by any transition operator T is metrically closed, a subspace, and a sublattice.*

Proof: Since T is continuous, the set is metrically closed; since T is linear, it is a subspace. Since T carries upper (lower) bounds into upper (lower)

* For a detailed discussion of this flow, cf. G. D. Birkhoff's *Dynamical Systems*, American Mathematical Society Colloquium Publications, vol. 9, New York, 1927. The ideas go back to Poincaré's *Méthodes Nouvelles de la Mécanique Céleste*, Paris, 1890; the proof of the invariance of volume, to Liouville.

† It can also be motivated directly. Additivity is plausible since hypothesis p with frequency λ and hypothesis q with frequency μ should yield inference pT with frequency λ and inference qT with frequency μ; that is, we should have $(\lambda p + \mu q)T = \lambda pT + \mu qT$. That (admissible) distributions must go into distributions is even more plausible heuristically.

‡ Isometrien resp. Verkürzungen in the sense of *Dehnungen, Verkürzungen, Isometrien*, by H. Freudenthal and W. Hurewicz, Fund. Math., 26 (1936), 120–2.

bounds, and is a contraction, it carries the unique upper bound $x = f \cup g$ to f and g satisfying $|f - x| + |x - g| \leqq |f - g|$, into itself.

It is a corollary that the "stable distributions" (the intersections of this subspace and D) are a closed convex set, and the number of stable distributions is either zero, or one (the "metrically transitive" case*), or infinite. All three cases are possible, but the second is by far the most interesting.

Considerations of symmetry often enable one to find stable distributions. Thus in Poincaré's card-shuffling problem mentioned in Example 2 of §169 (cf. also §173), the distribution which assigns to each arrangement the probability $1/52!$ is stable. For it is invariant under every permutation, and so under every habit of shuffling.

Similarly, in the kinetic theory of gases, if one uses the $3n$-dimensional phase-space,† one gets rotational symmetry in the velocity coordinates. This is how one finds the theoretical velocity-distribution for a given total energy: it is the symmetrical stable distribution.

If one had the "metrically transitive" case (cf. G. D. Birkhoff and B. O. Koopman, infra), one would know (by the ergodic theorems proved later) that this was the actual time-average—but this conjecture has never been proved.

173. Cyclic semi-groups of transition operators. One may call successive transition operators "independent" when they combine by operator-multiplication. One can motivate this definition in the finite case as follows. Let S and T be successive shuffles of a pack of cards. The compound transition probability u_{ij} from state i to state j will be the sum of the probabilities $u(i, k, j)$ that the pack will pass from i to j through the various intermediate states k. And if S and T are *independent*, we will have‡ $u(i, k, j) = s_{ik}t_{kj}$, whence $\| u_{ij} \|$ is ST.

If in addition the law of transition is "temporally homogeneous" (the phrase is due to Kolmogoroff, op. cit.), we get a *cyclic semi-group* of transition operators.

This may be *discrete*, and consist of a single transition operator T and its powers§ $T^2, T^3, \cdots$. Or it may be *continuous*, and satisfy $T^rT^s = T^{r+s}$ for all positive real r, s.

* In the sense of G. D. Birkhoff and Paul Smith, *Structure analysis of surface transformations*, Jour. de Math., 7 (1928), p. 365.

† The axes for the different molecules are shortened in varying proportions to compensate for difference in mass.

‡ J. L. Coolidge, *An Introduction to Mathematical Probability*, Oxford Press, 1925, p. 18. In quantum mechanics, successive transition operators are not independent; velocity tends to be preserved. This is even true of Brownian movement over very small intervals of time, which explains the apparent paradox of infinite velocity.

§ If T is any transition operator, then all its powers are transition operators. The generation of continuous cyclic semi-groups by infinitesimal operators is very difficult to describe except for finite-dimensional spaces (L). For special cases cf. G. Birkhoff, *Product integration*, M. I. T. Jour., 16 (1937), p. 123; also E. Hille, *Notes on linear transformations* II, Annals of Math., 40 (1939), 1–47.

A discrete cyclic semi-group of transition operators is usually called a "Markoff chain."* An example is furnished by Poincaré's card-shuffling problem. Let a pack of cards be shuffled. If $T = \| t_{ij} \|$ expresses the probability that a single shuffle will transform state i into state j, then the matrix T^n expresses the corresponding probability for n shuffles.

Continuous cyclic semi-groups of transition operators are called by Kolmogoroff "temporally homogeneous stochastic processes." Examples 2–3 of §170 afford instances of them; thus T^r might be the heat-transformation due to conduction in r minutes.

174. Digression: semi-groups vs. groups. It is well-known that in classical dynamics the past, as well as the future, can be determined from the present. In fact, the equations of dynamics are reversible in time.† Conversely, if the inverse T^{-1} of a transition operator T exists and is itself a transition operator, then by Theorem 9.4 both are contractions, and so isometries.

In summary, the case of a *group* of transition operators is precisely the *deterministic* case of isometries. Hence if a semi-group of stochastic transition operators is embedded in a group, none of the adjoined operators are transition operators. But it does not imply that no such embedding is possible.

In fact, in many cases—characteristically, where transitions take place by occasional jumps‡—the transition operators *do* have inverses. Indeed, in the temporally homogeneous case, the semi-group formed by them consists of the positive powers of an infinitesimal transition operator. By adjoining the negative powers also, one gets the group (cf. *Product integration*, loc. cit.).§

On the other hand, in Example 2 of §170 (as was already observed by Kelvin‖), the transition operators involved do not have inverses, and no embedding in a group is possible.

175. Markoff's Theorem. Now let Σ be any space (AL), and let there be given a (discrete or continuous) semi-group of transition operators T^r on Σ. Assume also

Markoff's Hypothesis: For some r, there is a positive lower bound d to the transforms pT^r of distributions p.

THEOREM 9.6: *If Markoff's Hypothesis is satisfied, then there is a unique stable distribution p_0. Moreover the pT^k tend to p_0 uniformly, with the rapidity that the terms of a convergent geometrical progression tend to zero.*

* It describes the Brownian movement of a ball cascading through a network of nails; cf. Fréchet, op. cit.

† G. D. Birkhoff, *Dynamical Systems*, p. 27.

‡ This is the "Poisson case" of A. Khintchine, *Die asymptotische Gesetze der Wahrscheinlichkeitsrechnung*, Berlin, 1933, Ergebnisse series. It includes the phenomena of radioactive disintegration, recording of cosmic rays on a Geiger counter, etc. Khintchine does not discuss the group aspect.

§ It should be emphasized that the inverse of a transition operator is *not* the "probability of causes" involved in Bayes' Theorem, etc.

‖ J. C. Maxwell, *Theory of Heat*, London, 1872, p. 244.

Proof: First note that (1) if $f \geqq 0$, then since for some distribution p, $f = |f| p$, we have $fT^r \geqq |f| d$.

Now let p and q be given. Set $h = p \frown q$, $f = p - h$, $g = q - h$, $\mu = 1 - |h|$. Then clearly $|p - q| = f + g$, $|p - q| = |f| + |g| = 2\mu$ and by the additivity of T^r, $pT^r = hT^r + fT^r$, $qT^r = hT^r + gT^r$. Hence $|pT^r - qT^r| = |fT^r - gT^r|$. But by the additivity of norm,

$$|fT^r - gT^r| = |fT^r| + |gT^r| - 2|fT^r \frown gT^r|.$$

And $|fT^r| = |gT^r| = \mu$, whence by (1) $fT^r \frown gT^r \geqq \mu d$. Combining, $|pT^r - qT^r| \leqq 2\mu - 2\mu|d| = 2\mu(1 - |d|)$. We conclude

LEMMA 1: $|pT^r - qT^r| \leqq (1 - |d|)|p - q|$. *In words, T^r contracts D uniformly, by ratios at least* $1:1 - |d|$.

Theorem 9.6 is a consequence of this lemma and simple geometrical considerations.* Indeed, the operators T^{kr} contract D into a subset of itself, of diameter at most $2[1 - |d|]^k$. Hence for any q, the qT^s satisfy Cauchy's condition for convergence: $|qT^{kr} - qT^{kr+s}| \leqq 2[1 - |d|]^k$ for all $s \geqq 0$. Hence, D being a complete metric space, they converge to a limit p_0.

Furthermore, p_0 is *stable*. For since $qT^s \to p_0$ as $s \to \infty$, and T is continuous, $qT^sT = p_0T$; but $qT^{s+1} \to p_0$, and so $p_0T = p_0$. The *uniqueness* of p_0 and the convergence of all pT^s to p_0 now follow from the fact that the diameter of DT^{kr} is at most $2[1 - |d|]^k$, mentioned above.

COROLLARY 1: Sup $pT^m \leqq p + \sum_{k=0}^{\infty} (pT - p)T^k$ *is finite, for any fixed p, in the case of a discrete semi-group.*†

COROLLARY 2: *Let $T_1, \cdots, T_n$ be any sequence of transition operators, and denote* $\inf_{p \in D} pT_i$ *by d_i. Then $T_1T_2 \cdots T_n$ induces a uniform contraction on D, shrinking distances in ratios at least* $1:\prod_{i=1}^{n} (1 - |d_i|)$.

Incidentally, it is clear that in the finite-dimensional case the conclusion of Theorem 9.6 conversely implies Markoff's hypothesis—hence so does Lemma 1, the condition of uniform contraction.

176. Ergodic theorems. It is clear that in the deterministic (isometric) case of classical mechanics, the transforms pT^r of non-stable distributions cannot converge, since $|pT^{r+1} - pT^r| = |pT - p|$ identically.

* Historical note: Theorem 9.6 was first proved for finite matrices by G. Frobenius, *Über Matrizen aus nicht negativen Elementen*, S.-B. Berlin (1912), 456–66. The geometrical half of the proof goes back to C. Neumann, who used it to solve Dirichlet's problem for convex regions (cf. Picard's *Traité d'Analyse*, second ed., vol. 1, p. 170). It is used by Picard (op. cit., vol. 2, p. 301) to prove the existence of solutions of non-linear differential equations. The geometrical approach to the theorem on matrices was sketched by G. Rajchmann (Comptes Rendus, 190 (1930), p. 729; cf. also J. Hadamard, Atti Congresso Bologna, 5 (1928), 133–9). The general case was first handled by the author [8].

† In the continuous case, this is still true if the pT^s are bounded on any interval $0 \leqq s \leqq s_0$, for a similar reason.

Nevertheless, as is well-known, their *averages* p_N ($\sum_{k=1}^{N-1} pT^k/N$ in the discrete, $\int_0^N pT^s ds/N$ in the continuous case) do converge; theorems proving this conclusion are usually called "ergodic theorems."*

The fact that Markoff's Theorem 9.6 asserts much more than ergodicity suggests that one might prove ergodic theorems covering simultaneously the deterministic and the stochastic cases. The remaining sections will be devoted to such generalized ergodic theorems; the main results will be summarized now.

Let $\{T^r\}$ be any discrete or continuous cyclic semi-group of linear operators on a Banach space. We shall call an element f of the space "ergodic" if and only if the means† g_s of the fT^r converge metrically to a fix-point.

Our theorems will apply to the case where the T^r are *isometries* or *contractions*; this includes the case of transition operators, by Theorem 9.4.

Not all elements need to be ergodic. For example, let T be the operator $[x_1, x_2, x_3, \cdots] \to [0, x_1, x_2, \cdots]$ on the space (1). Then $f = [1, 0, 0, \cdots]$ is not ergodic; in fact $|g_{2s} - g_s| > 1/2$ for all s.‡

We prove first that if the g_s lie in a "weakly compact" set, then f is ergodic. From this we deduce as a corollary: if the space is finite-dimensional or the space (L_p) $[p > 1]$, then every element is ergodic. Further, in the space (L), if the fT^r are bounded (in the lattice-theoretic sense), then f is ergodic; while if $f(x) = 1$ is invariant under $\{T^r\}$, then every element is ergodic.

Using a remark of Kakutani, the last results are then extended to all spaces (AL). It is shown that even in this case, if the fT^r are bounded, then f is ergodic —and that the same conclusion holds if $0 < f < a$, where a is ergodic.§

177. General ergodic theorems. Following the program just outlined, we begin by observing the elementary

THEOREM 9.7: *The set E of elements ergodic under any cyclic semi-group $\{T^r\}$ of linear isometries or contractions of any Banach space is a metrically closed subspace. It contains all its images and antecedents under $\{T^r\}$.*

* Ergodic theorems have a philosophical bearing on statistical mechanics, but the connection between the two is still incomplete. For a historical discussion of them, cf. G. D. Birkhoff and B. O. Koopman, *Recent contributions to the ergodic theory*, Proc. Nat. Acad. Sci., 18 (1932), 279–82; also E. Hopf's *Ergodentheorie*, Berlin, 1937, and N. Wiener, *The ergodic theorem*, Duke Jour., 5 (1939), 1–18. For a sharper theorem, applying to individual path-curves, and hence more applicable to qualitative dynamics, cf. G. D. Birkhoff's *Proof of the ergodic theorem*, Proc. Nat. Acad. Sci., 17 (1931), 656–60. The proof of this is, incidentally, combinatory and so lattice-theoretic in spirit.

† I.e., $(1/s) \sum_{r=0}^{s-1} fT^r$ in the discrete, and $(1/s) \int_0^s fT^r\, dr$ in the continuous case.

‡ The g_s do not even converge weakly relative to the functional $\lambda(x) = \sum_{i=1}^{\infty} x_i \cos(\log\log(i + 100))$.

§ Historical note: Cf. Fréchet, op. cit., p. 109. The general case was first treated by the author [8]. His result was greatly improved by F. Riesz, *Some mean ergodic theorems*, Jour. Lond. Math. Soc., 13 (1938), 274–8, and by S. Kakutani and K. Yosida, *Operator-theoretical treatment of Markoff's process*, Proc. Imp. Acad. Tokyo, 14 (1938), pp. 286, 333, 363. Kakutani's extension to spaces (AL) was published in *Mean ergodic theorems in abstract (L)-spaces*, Proc. Imp. Acad. Tokyo, 15 (1939), 121–3. The proof given below is based on the method of Riesz.

Proof: If the means g_s of f and g_s^* of f^* converge to a and a^* respectively, then the means $g_s + g_s^*$ resp. λg_s of $f + f^*$ resp. λf, converge to $a + a^*$ resp. λa; hence E is a subspace. It is equally easy to show that E is metrically closed. For $|f - f^*| < \epsilon$ implies $|g_s - g_s^*| < \epsilon$ for all s. Hence if $|f^{(n)} - f^*| \to 0$ as $n \to \infty$, and every $f^{(n)}$ is ergodic, the $a^{(n)} = \lim_{s\to\infty} g_s^{(n)}$ converge metrically to a limit a. Moreover

$$|g_s^* - a| \leqq |g_s^* - g_s^{(n)}| + |g_s^{(n)} - a^{(n)}| + |a^{(n)} - a|.$$

But the first and third of these terms are bounded by $|f^* - f^{(n)}|$ and so are arbitrarily small when n is large. While for fixed n, the second tends to zero as $s \to \infty$. Hence $g_s^* \to a$ as $s \to \infty$, and f^* is ergodic. Finally, that E contains all its images and antecedents under $\{T^r\}$ is obvious, since every fT^r has the same limit of means (if any) as f.

But any metrically closed subspace of a Banach space is ipso facto weakly closed,† hence

Corollary: *The set E is closed under the weak topology of the space.*

Theorem 9.8: *If the means g_s of the fT^r lie in a weakly compact set, then f is ergodic.*

Proof: By hypothesis, some subsequence $\{g_{s(i)}\}$ of $\{g_s\}$ converges weakly to a limit a. But every $f - fT^r$ is ergodic, since if $s > r$,

$$\left|\frac{1}{s}\sum_{k=0}^{s-1}(f - fT^r)T^k\right| = \frac{1}{s}\left|\sum_{k=0}^{r-1} fT^k - \sum_{k=s}^{s+r-1} fT^k\right| \leqq 2r/s,$$

so that the means of its transforms converge metrically to 0. (A similar formula holds for continuous semi-groups.)

Hence every difference $f - g_{s(i)}$, being a mean of $f - fT^r$, is in the set E of ergodic elements. But this set is weakly closed (by the Corollary of Theorem 9.7); hence $f - a$ is in E. It remains to show that a is in E—whence $f = (f - a) + a$ will be in E. But this is an obvious corollary of the

Lemma: *If any subsequence $\{g_{s(i)}\}$ of the g_s converges metrically or weakly to a limit a, then a is a fix-point.*

Proof: One can assume $|f| = 1$ without losing generality. But in this case, for any positive integer s and any $k < s$, $s(g_s - g_sT^k)$ is $\sum_{r=0}^{k-1} fT^r - \sum_{s=k}^{s-1} fT^r$ resp. $\int_0^k fT^r dr - \int_{s-k}^s fT^r dr$ in the discrete resp. continuous case. Hence

$$(1) \qquad |g_s - g_sT^k| \leqq 2k/s.$$

Now with metric convergence, $|g_{s(i)} - a| \to 0$ implies $|g_{s(i)}T^k - aT^k| \to 0$ for any k. But $|g_{s(i)} - g_{s(i)}T^k| \to 0$ by (1); hence $|a - aT^k|$ is less than any

† According to Banach [1], p. 133, $x_i \to x$ "weakly" if and only if for every (bounded) linear functional λ, $\lambda(x_i) \to \lambda(x)$.

The result quoted is one of Banach's fundamental theorems; a complete proof is given by Banach [1], pp. 53–8 and 113–4.

positive constant, and $a = aT^k$. Similarly, with weak convergence, $\lambda(g_{s(i)} - a) \to 0$ for all λ implies $\lambda(g_{s(i)}T^k - aT^k) \to 0$ for all λ and k. But by (1), $\lambda(g_{s(i)} - g_{s(i)}T^k) \to 0$ for all λ and k; hence for all λ, $\lambda(a - aT^k)$ is less than any positive constant, and $a = aT^k$.

Whether or not the theorem holds whenever the T^r are continuous linear operators is an undecided question.

178. Corollaries. From the fundamental Theorem 9.8 one can infer directly,

Corollary 1: *Let $\{T^r\}$ be any cyclic semi-group of linear isometries or contractions of a Banach space which is finite-dimensional, a space (L^p), or a space (l^p) $(p > 1)$. Then every element is ergodic under $\{T^r\}$.*

For in the finite-dimensional case the unit sphere is metrically compact; and in the other cases the unit sphere is weakly compact. A direct proof can also be given in the case of the spaces (L^p) and (l^p) $(p > 1)$.*

Corollary 2: *If $a \leqq fT^r \leqq b$ for all r, and the Banach space is the space (L), then f is ergodic.*

In other words, f is ergodic if its transforms are bounded lattice-theoretically. Proof: the set $a \leqq f \leqq b$ is well-known to be weakly compact. The same argument applies to the space (l).

Corollary 3: *If the function $f(x) = 1$ is invariant under transition operators T^r on the space (L), then every element is ergodic.*

For the condition $a \leqq f(x) \leqq b$ for all x is preserved under $\{T^r\}$ (transition operators being linear and positive); hence Corollary 2 applies to show that all bounded functions are ergodic. But these are topologically dense; hence the argument is completed by Theorem 9.7.

179. Extension to spaces (AL). We are now ready to sketch the extension to arbitrary spaces (AL). This can be based, as Kakutani has pointed out,† on the fact that the space (L) is a *universal* separable space (AL). More precisely, any separable (AL) is simultaneously isometric, linearly isomorphic and lattice-isomorphic with a closed linear subspace of the space (L). This was essentially proved by H. Freudenthal [1]; we omit the proof.

But, at least if $\{T^r\}$ is a discrete semi-group, any set of fT^r generates a *separable* subspace of the space (AL) containing it, when its elements are combined linearly and lattice-theoretically. We infer that Corollary 2 applies to arbitrary spaces (AL); whence

Theorem 9.9: *Let T be any transition operator on any space (AL). Any element whose transforms under the iterates of T are bounded lattice-theoretically is ergodic.*

* Cf. the author's *The mean ergodic theorem*, Duke Jour., 5 (1939), 19-20. For the properties of the spaces considered cf. Banach [1], pp. 84, 130-1.

† *Mean ergodic theorem in abstract (L)-spaces*, Proc. Imp. Acad. Tokyo, 15 (1939), 121-3.

THEOREM 9.10: *Let T be as in Theorem 9.9. If f is ergodic and $0 < x < f$, then x is ergodic.*

Proof: Let y_s and g_s denote the sth means of x and f respectively; since T is positive, $0 < y_s < g_s$ for all s. But by hypothesis $\{g_s\}$ converges to a stable distribution a as $s \to \infty$; hence

$$| y_s - (y_s \frown a) | \leqq | g_s - (g_s \frown a) | \to 0.$$

By Theorem 9.9, $y_s \frown a$ is ergodic for all s; hence the "ergodic oscillation" $\limsup_{s,s' \to \infty} | y_s - y_{s'} |$ of x is bounded by all positive numbers. For it is equal to that of every y_s (since every $x - y_s$ is ergodic as in the proof of Theorem 9.8) and the ergodic oscillation of y_s is bounded by $| y_s - y_s \frown a |$ since $y_s \frown a$ is ergodic. Hence x is ergodic.

Remark 1: Theorem 9.10 applies to continuous cyclic semi-groups in the case of the space (L); to prove this, simply change the reference to Theorem 9.9 into one to Corollary 2 of Theorem 9.8.

Remark 2: Theorems 9.7 and 9.10 suggest that the ergodic elements of a space (AL) always form a *normal* subspace. But this conjecture can easily be disproved by an example. Let T be the operator $[x_1, x_2, x_3, \cdots] \to [0, x_1, x_2, \cdots]$ on the space (1) described in §176. Then $f = [1, -1, 0, \cdots]$ is ergodic, whereas f^+ and f^- are not.

UNSOLVED PROBLEMS

There are a number of unsolved problems in lattice theory, to which more or less attention has been devoted. They are listed here, in the hopes that future efforts may yield their answers.

(1) The free modular lattice with $n \geqq 4$ generators has never been exactly described: no general rule is known, deciding in a finite number of steps when two expressions necessarily represent the same element in a modular lattice.

(2) How many elements are there in the free distributive lattice with n generators? (Dedekind.)

(3) Is the completion by cuts of a modular lattice necessarily modular? of a distributive lattice necessarily distributive? (MacNeille.)

(4) Is any lattice, each of whose elements has one and only one complement, necessarily a Boolean algebra? This is known to be true in the complete, atomistic case.

(5) Describe the most general completely distributive lattice, in a way analogous to Tarski's description of the most general completely distributive Boolean algebra. This should be feasible by direct methods.

(6) Enumerate and classify all finite-dimensional vector lattices. It is known that there is one and only one *σ-complete* vector lattice of each finite dimension.

(7) Find a necessary and sufficient condition that in a lattice the *number* of irreducible components in representations of any element as a meet of meet-irreducible elements should be unique. When is this true both for meets and joins? Assume finiteness if it helps. A condition that the components themselves be unique has been found by Dilworth.

(8) Prove that the representation of a finite partially ordered system as the product of indecomposable factors is unique to within pairwise isomorphism of the factors. This is known in the presence of a O and I; also in the case of cardinal numbers.

(9) No complete enumeration of all finite non-Desarguesian plane geometries is known; this is a very difficult problem.

(10) Find a neat necessary and sufficient condition that a partially ordered semi-linear space (such as that of convex functions) be embeddable (a) in a vector lattice, and (b) in a partially ordered linear space.

(11) Is every complete lattice topologically bicompact? Is it true that if the intersection of any family of subsets of a complete lattice is void, and if the subsets are closed relative to Moore-Smith convergence, then there exists a finite subfamily having a void intersection?

There are many undecided questions regarding the possibility of representing lattices in various ways. A number of these will be stated below.

(12) Which modular lattices can be represented as lattices of subgroups? of subspaces of linear spaces? Do such lattices satisfy any identities besides the modular identity?

(13) Which lattices can be represented by lattices of partitions? Does this

class include all lattices? all modular lattices? Is it self-dual? Do its members satisfy any lattice identity not deducible from L1–L4? It is not even known whether the dual of the lattice of all partitions of four elements can be represented as a lattice of partitions.

(14) One can ask the same questions regarding lattices of convex bodies (Stone, Colloquium Lectures, 1939).

(15) What lattices can be represented as sublattices of matroid (i.e., exchange) lattices? Does this class include all lattices? Is it self-dual?

(16) Which lattices are isomorphic with sublattices of complemented lattices? of relatively complemented lattices? Which modular lattices can be represented as sublattices of complemented modular lattices? (It is known that every distributive lattice can be embedded in a Boolean algebra, which suggests that they all can.)

(17) Every Boolean algebra with more than two elements has an automorphism besides the identity. Try the countable case.

BIBLIOGRAPHY

This bibliography is not in any sense complete; it comprises only those works most frequently referred to in the text. More complete references will be found in the text. For aspects relating to logic, a complete bibliography has been published by the Journal of Symbolic Logic.

A. A. ALBERT, [1] *Modern Higher Algebra*, Chicago, 1937.

P. ALEXANDROFF AND H. HOPF, [1] *Topologie*, Berlin, 1935.

A. A. BENNETT, [1] *Semi-serial order*, Am. Math. Monthly, 37 (1930), 418–23.

G. BERGMANN, [1] *Zur Axiomatik der Elementargeometrie*, Monats. f. Math. u. Phys., 36 (1929), 269–84.

G. BIRKHOFF, [1] *On the combination of subalgebras*, Proc. Camb. Phil. Soc., 29 (1933), 441–64; [2] *Applications of lattice algebra*, ibid., 30 (1934), 115–22; [3] *On the lattice theory of ideals*, Bull. Am. Math. Soc., 40 (1934), 613–9; [4] *Combinatorial relations in projective geometries*, Annals of Math., 36 (1935), 743–8; [5] *Abstract linear dependence and lattices*, Am. Jour., 57 (1935), 800–4; [6] *On the structure of abstract algebras*, Proc. Camb. Phil. Soc., 31 (1935), 433–54; [7] *On rings of sets*, Duke Jour., 3 (1937), 443–54; [8] *Dependent probabilities and spaces (L)*, Proc. Nat. Acad. Sci., 24 (1938), 154–9.

G. BOOLE, [1] *The mathematical analysis of logic*, Cambridge, 1847; [2] *An investigation into the laws of thought*, London, 1854.

R. DEDEKIND, [1] *Über Zerlegungen von Zahlen durch ihre grössten gemeinsamen Teiler*, Festschrift Techn. Hoch., Braunschweig (1897), and *Gesammelte Werke*, vol. 2, pp. 103–48; [2] *Über die von drei Moduln erzeugte Dualgruppe*, Math. Ann., 53 (1900), 371–403, and *Gesammelte Werke*, vol. 2, pp. 236–71.

H. FREUDENTHAL, [1] *Teilweise geordnete Moduln*, Proc. Akad. Wet. Amsterdam, 39 (1936), 641–51.

V. GLIVENKO, [1] *Géométrie des systèmes de choses normées*, Am. Jour., 58 (1936), 799–828; [2] *Théorie des Structures*, Paris, 1938.

F. HAUSDORFF, [1] *Grundzüge der Mengenlehre*, Leipzig, 1914, second edition of *Mengenlehre*, Berlin, 1927. References are to the second edition unless otherwise specified.

E. V. HUNTINGTON, [1] *Sets of independent postulates for the algebra of logic*, Trans. Am. Math. Soc., 5 (1904), 288–309.

L. KANTOROVITCH, [1] *Lineare halbgeordnete Räume*, Math. Sbornik, 2 (44) (1937), 121–68.

FRITZ KLEIN, [1] *Einige distributive Systeme in Mathematik und Logik*, Jahr. D. M.-V., 38 (1929), 35–40; [2] *Zur Theorie der abstrakten Verknüpfungen*, Math. Ann., 105 (1931), 308–23; [3] *Grundzüge der Theorie der Verbände*, Math. Ann., 111 (1935), 596–628.

G. KÖTHE, [1] *Die Theorie der Verbände, ein neuer Versuch Zur Grundlegung der Algebra und der projektiven Geometrie*, Jahr. D. M.-V., 47 (1937), 125–44.

A. KUROSCH, [1] *Durchschnittsdarstellungen mit irreduziblen Komponenten in Ringen und in sogenannten Dualgruppen*, Math. Sbornik, 42 (1935), 613–16.

H. MACNEILLE, [1] Harvard Doctoral Thesis (unpublished), 1935; [2] *Partially ordered sets*, Trans. Am. Math. Soc., 42 (1937), 416–60.

K. MENGER, [1] Ergebnisse eines Kolloquiums, Vienna, vol. 1, p. 28; [2] *Bemerkungen zu Grundlagenfragen*, IV, Jahr. D. M.-V., 37 (1928), 309–13; [3] *New foundations of projective and affine geometry*, Annals of Math., 37 (1936), 456–82.

GR. C. MOISIL, [1] *Recherches sur l'algèbre de la logique*, Annales Sci. de Jassy, 22 (1936), 1–118.

E. H. Moore, [1] *The New Haven Mathematical Colloquium, Introduction to a form of general analysis*, New Haven, 1910.

J. von Neumann, [1] *Continuous geometries*, and *Examples of continuous geometries*, Proc. Nat. Acad. Sci., 22 (1936), 92–109; [2] *Lectures on Continuous Geometries*, Princeton, 1936–7, 2 vols.; [3] *On regular rings*, Proc. Nat. Acad. Sci., 22 (1936), 707–13.

O. Ore, [1] *On the foundations of abstract algebra*, I, Annals of Math., 36 (1935), 406–37; [2] *On the foundations of abstract algebra*, II, ibid., 37 (1936), 265–92.

C. S. Peirce, [1] *On the algebra of logic*, Am. Jour., 3 (1880), 15–57; [2] *On the algebra of logic*, ibid., 7 (1884), 180–202.

Fr. Riesz, [1] *Sur la décomposition des opérations fonctionelles*, Atti Congresso Bologna, 3 (1928), 143–8; [2] *Sur . . . la théorie générale des opérations linéaires*, Annals of Math., 41 (1940), 174–206.

E. Schröder, [1] *Algebra der Logik*, 3 vols., Leipzig, 1890–5, with bibliography at end of each volume.

Th. Skolem, [1] *Om konstitutionen av den identiske kalkuls grupper*, Third Scand. Math. Congr. (1913), 149–63.

M. H. Stone, [1] *Boolean algebras and their application to topology*, Proc. Nat. Acad. Sci., 20 (1934), 197–202; [2] *Subsumption of Boolean algebras under the theory of rings*, ibid., 21 (1935), 103–5; [3] *The theory of representations for Boolean algebras*, Trans. Am. Math. Soc., 40 (1936), 37–111; [4] *Applications of the theory of Boolean rings to general topology*, ibid., 41 (1937), 375–481; [5] *Topological representations of distributive lattices and Brouwerian logics*, Cas. Mat. Fys., 67 (1937), 1–25.

A. Tarski, [1] *Sur les classes d'ensembles closes par rapport à certaines opérations élémentaires*, Fund. Math., 16 (1929), 181–305; [2] *Zur Grundlegung der Boole'schen Algebra*, I, ibid., 24 (1935), 177–98; [3] *Grundzüge des Systemenkalkuls*, ibid., 25 (1936), 503–26, and 26 (1936), 283–301.

O. Veblen and W. H. Young, [1] *Projective Geometry*, Boston, 1910.

B. L. van der Waerden, [1] *Moderne Algebra*, Berlin, 1930–1, 2 vols.

H. Wallman, [1] *Lattices and topological spaces*, Annals of Math., 39 (1938), 112–26.

A. N. Whitehead, [1] *Universal Algebra*, Cambridge, 1898.

H. Whitney, [1] *The abstract properties of linear dependence*, Am. Jour., 57 (1935), 507–33.

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